Chapter 4

Bivariate discrete distribution

4.1 Introduction

In this chapter we consider pairs of discrete random variables which arise from one or more experiments. We are interested in the joint distribution of X and Y. This might be specified as a joint probability function P(X-x, Y= y) or as a table of such joint probabilities. We make a number of definitions and then illustrate the idea with an example.

Definition: A joint probability function *P(X=x, Y=y)* satisfies *P(X=x, Y=y)* ≥ 0 and

$$\sum_x \sum_y P(X = x, Y = y) = 1$$

Definition: The marginal distribution of X is defined by the probability function

$$P(X = x) \sum_y P(x + x, Y = y)$$

Note that P(X=x) ≥ 0 and $\sum_x P(X = x) = 1$. The mean and variance of X can be defined in the usual way.

Definition: The conditional distribution of X given Y =y is defined by the probability function

$$P(X = x / Y = y) = \frac{P(X = x, Y = y)}{P(Y = y)}$$

The conditional means of X given Y=y is defined by

$$E[X / Y = y)] = \sum_x xP(X / Y = y)$$

and similarly for the variance.

Definition: The covariance of *X* and *Y* is defined by

$$Cov(X, Y) = E[XY] - E(X]E[Y]$$

where

$$E[XY] = \sum_x \sum_y xyP(X = x, Y = y)$$

Definition: The correlation between X and Y is defined by

$$\frac{Cov[X, Y]}{\sqrt{Var[X]Var[Y]}}$$

The correlation always lies between –1 and +1

Definition: Random variables X and Y are said to be independent if

$$P(X = x, Y = y) = P(X = x)P(y = y)$$

If X and Y are independent then *Cov [X,Y]=0*. The covariance is NOT true. There exist many pairs of random variables with *Cov[X,Y]=0* which are not independent.

We now give an example which illustrate these ideas.

Example: The random variables *X* and *Y* have a joint probability function given by

$$f(x,y) = \begin{cases} c(x^2 y + x) & x = -2,-1,0,1,2 \quad y = 1,2,3 \\ 0 & \text{otherwise} \end{cases}$$

Determine the value of c.
Find $P(X > 0)$ and $P(X + Y = 0)$
Find the marginal distribution of X and Y
Find *E[X]* and *V[X]*
Find *E[Y]* and *V[Y]*
Find the conditional distribution of *X* given *Y=1* and *E[X| Y=1]*
Find the probability function for *Z = X + Y* and show that *E [Z] = E[X] + E[Y]*
Find *Cov[X,Y]* and show that *Var[Z]=Var[X] + Var [Y] + 2 Cov [X, Y]*
Find the correlation between *X* and *Y*.
Are *X* and *Y* independent?

It is easiest with this sort of problem to find the table of the joint probability function. In this case we obtain

		X					
		-2	-1	0	1	2	
	1	2c	0	0	2c	6c	10c
Y	2	6c	c	0	3c	10c	20c
	3	10c	2c	0	4c	14c	30c
		18c	3c	0	9c	30c	60c

Margins are also included in the table. Since the probability must add to one, c must equal 1/60. From the table *P(X > 0)* = 39/60 and *P(X+Y=0) = P(X=-2, Y = 2) + P (X+-1, Y=1)* = 1/10

From the margins in the table we obtain the marginal distributions of *X* and *Y*. These are *P (X=-2) = 18/60 , P(X = -1) = 3/60 , P(X=0)= 0, P(X=1) = 9/60, P(X=2) = 30 /60* and *P(Y=1)=1/6, P(Y=2)=1/3, P(Y=3) = ½* , note that the probabilities in these marginal distributions sum to one as they should.
Now

$$E[X] = -2 \times 18/60 + 1 \times 3/60 + 0 \times 0 + 1 \times 9/60 + 2 \times 30/60 = 30/60 = 1/2$$

and

$$E[X^2] = 4 \times 18/60 + 1 \times 3/60 + 0 \times 0 + 1 \times 9/60 + 4 \times 30/60 = 204/60 = 3.4$$

Thus $Var[X] = 3.4 - 0.5^2 = 3.15$
Also

$$E[Y] = 1 \times 1/6 + 2 \times 1/3 + 3 \times 1/2 = 14/6 = 7/3$$

$$E[Y^2] = 1 \times 1/6 + 4 \times 1/3 + 9 \times 1/2 = 36/6 = 6.0$$

thus $Var[X] = 6.0 - (7/3\}^2 = 5/9$
Given that *Y =1* the conditional distribution of X is *P(X=-2 | Y=1)* = 2c/10c = 0.2,

P(X = -1 | Y=1) = 0, P(X =0/Y=1) = 0, P(X = 1/Y =1) 2c/10c = 0.2, P (X= 2 | Y = 1) = 6c/10c = 0.6. Again not that the probabilities sum to one as they should.

Now

$$E[X \mid Y = 1] = -2 \times 0.2 + -1 \times 0 + 0 \times 0 \times +1 \times 0.2 + 2 \times 0.6 = 1..$$

The Probability function for *Z* is given by

Z	-1	0	1	2	3	4	5
P(Z=z}	2/60	6/60	11/60	4/60	9/60	14/60	14/60

Thus

$$E[Z] = \frac{1}{60}(-1 \times 2 + 1 \times 11 + 2 \times 4 + 3 \times 9 + 4 \times 14 + 5 \times 14)$$

$$= \frac{170}{60}$$

$$= 2\frac{5}{6}$$

$$= \frac{1}{2} + 2\frac{1}{3} = E[X] + E[Y]$$

To Find *Cov [X , Y]* we see that *E[XY] = 1* and hence *Cov [X , Y] = E[XY] – E[X]*. E[Y] = $-\frac{1}{6}$. Also $E[Z^2]$ is given by

$$E[Z^2] = \frac{1}{60}(1 \times 2 + 1 \times 11 + 4 \times 4 + 9 \times 9 + 16 \times 14 + 25 \times 14) = \frac{684}{60}$$

Thus *Var[Z] =* $\frac{684}{60} - \left(\frac{170}{60}\right)^2 = \frac{3035}{900} = 3.3722.$ Now *Var [X] + Var[Y] + 2 Cov { X, Y] =* $3.15 + \frac{5}{9} - 2 \times \frac{1}{6} = 3.3722$

The correlation between X and Y is given by $Cov[X,Y]/\sqrt{Var[X]Var[Y]}$ which is $(-1/6)/\sqrt{3.15 \times 5/9} = -0.126$

X and *Y* are independent. *Cov [X,Y]* $\neq 0$ and *P(X = ⁻1 , Y = 1)* $\neq$ *P(X = ⁻1) P(Y=1)*, for example

Example: The following experiment is carried out. Three fair coins are tossed. Any coins showing heads are removed and the remaining coins are tossed. Let *X* be the number of heads on the first toss and Y the number of heads on the second toss. Now that if *X = 3* then Y = 0. Find the joint probability function and marginal distribution of *X* and *Y*.

Note that *P(Y=y, X=x) = P(Y=y|X=x)P(X=x)*. Suppose *X =0*, this has probability 0.5^3. Then *Y | X = 0* has a binomial distribution with parameters n = 3 and p = 0.5. Similarly *Y | X = 1* has a binomial distribution with parameters *n = 2* and *p = 0.5*. In this way we see that the table of joint distribution is given by

		X				
		0	1	2	3	
	0	1/64	6/64	12/64	8/64	27/64
Y	1	3/64	12/64	12/64	0	27/64
	2	3/64	6/64	0	0	9/64
	3	1/64	0	0	0	1/64
		1/8	3/8	3/8	1/8	1

The marginal distribution for *X* and *Y* are given in the table. We see that *X* is binomial with parameters $n = 3$ and $p = 0.5$.

4.2 Exerxise

1. Random variables *X* and *Y* have a joint distribution given by the following table

		Y			
		1	2	3	4
	1	0	.1	.15	.05
X	2	.2	.05	.05	.1
	3	.1	.15	0	.05

   Find $E[X]$ and $E[X|Y=2]$

2. Show that the following table give the valid joint probability function for the discrete random variables X and Y.

		Y		
		1	2	3
	1	0.20	0.09	0.01
X	2	0.04	0.30	0.06
	3	0.01	0.06	0.2

   Find the following
   (i) $P(X + Y = 4)$
   (ii) The marginal probability function of *X*
   (iii) $E[X]$
   (iv) $Var[X]$
   (v) $E[Y \mid X = 2]$
   (vi) $Cov[X,Y]$
   (vii) The correlation between *X* and *Y*

   Find a joint probability function with the same marginal probability functions for *X* and *Y* but such that *X* and *Y* are independent.