Transportation and Assignment Models

Learning Objectives

Students will be able to:

1. Structure special LP problems using the transportation and assignment models.
2. Use the N.W. corner, VAM, MODI, and stepping-stone method.
3. Solve facility location and other application problems with transportation methods.
4. Solve assignment problems with the Hungarian (matrix reduction) method.

Introduction

Two Special LP Models

The *Transportation* and *Assignment* problems are types of LP techniques called *network flow problems*.

1. Transportation Problem
   - Deals with the distribution of goods from several points of supply (*sources*) to a number of points of demand (*destinations*).
   - Transportation models can also be used when a firm is trying to decide where to locate a new facility.
   - Good financial decisions concerning facility location also attempt to minimize total transportation and production costs for the entire system.

Introduction

Two Special LP Models

2. Assignment Problem
   - Refers to the class of LP problems that involve determining the most efficient assignment of
     - people to projects,
     - salespeople to territories,
     - contracts to bidders,
     - jobs to machines, etc.
   - The objective is most often to minimize total costs or total time of performing the tasks at hand.
   - One important characteristic of assignment problems is that **only one** job or worker is assigned to one machine or project.

Importance of Special- Purpose Algorithms

Special-purpose algorithms (more efficient than LP) exist for solving the Transportation and Assignment problems.

- As in the simplex algorithm, they involve
  - finding an initial solution,
  - testing this solution to see if it is optimal, and
  - developing an improved solution.
  - repeating these steps until an optimal solution is reached.
- The Transportation and Assignment methods are much simpler than the simplex algorithm in terms of computation.

Importance of Special- Purpose Algorithms

Streamlined versions of the simplex method are important for two reasons:

1. Their computation times are generally 100 times faster than the simplex algorithm.
2. They require less computer memory (and hence can permit larger problems to be solved).

Importance of Special- Purpose Algorithms

- Two common techniques for developing initial solutions are:
  - the *northwest corner method* and
  - *Vogel's approximation method.*
- After an initial solution is developed, it must be evaluated by either
  - *the stepping-stone method* or
  - *the modified distribution (MODI) method.*
- Also introduced is a solution procedure for assignment problems alternatively called
  - the *Hungarian method,*
  - *Flood's technique*, or
  - the *reduced matrix method*.

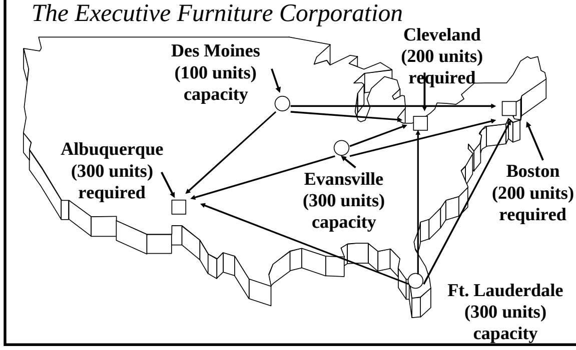
Setting Up a Transportation Problem

The Executive Furniture Corporation

- Manufactures office desks at three locations:
  - Des Moines, Evansville, and Fort Lauderdale.
- The firm distributes the desks through regional warehouses located in
  - Boston, Albuquerque, and Cleveland (see following slide).

Transportation Problem

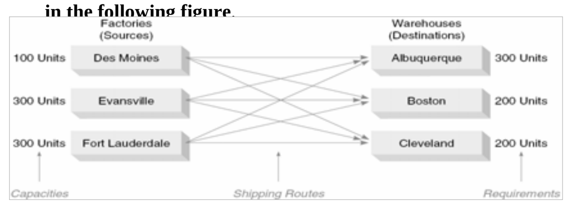
The Executive Furniture Corporation



Setting Up a Transportation Problem

The Executive Furniture Corporation

- **An estimate of the monthly production capacity at each factory and an estimate of the number of desks that are needed each month at each of the three warehouses is shown in the following figure.**



Transportation Costs

The Executive Furniture Corporation

- **Production costs per desk are identical at each factory; the only relevant costs are those of shipping from each *source* to each *destination*.**
- **These costs are shown below.**
- **They are assumed to be constant regardless of the volume shipped.**

From (Sources)	*To (Destinations)* Albuquerque	Boston	Cleveland
Des Moines	$5	$4	$3
Evansville	$8	$4	$3
Fort Lauderdale	$9	$7	$5

Transportation Costs

The Executive Furniture Corporation

1. The first step is to set up a *transportation table.*
   ¤ Its purpose is to summarize concisely and conveniently all relevant data and to keep track of algorithm computations.
   * It serves the same role that the simplex tableau did for LP problems.
2. Construct a transportation table and label its various components.
   ¤ Several iterations of table development are shown in the following slides.

Unit Shipping Cost:1Unit, Factory to Warehouse

The Executive Furniture Corporation

	Albuquerque (A)	Boston (B)	Cleveland (C)	Factory Capacity
Des Moines (D)	5	4	3	
Evansville (E)	8	4	3	
Fort Lauderdale (F)	9	7	5	
Warehouse Req.				

Cell representing a source-to-destination assignment

Cost of shipping 1 unit from Fort Lauderdale factory to Boston warehouse

Total Demand and Total Supply

The Executive Furniture Corporation

	Albuquerque (A)	Boston (B)	Cleveland (C)	Factory Capacity
Des Moines (D)				100
Evansville (E)				300
Fort Lauderdale (F)				300
Warehouse Req.	300	200	200	700

Transportation Table for Executive Furniture Corp.

The Executive Furniture Corporation

	Albuquerque (A)	Boston (B)	Cleveland (C)	Factory Capacity
Des Moines (D)	5	4	3	100
Evansville (E)	8	4	3	300
Fort Lauderdale (F)	9	7	5	300
Warehouse Req.	300	200	200	700

Initial Solution Using the Northwest Corner Rule

Start in the upper left-hand cell and allocate units to shipping routes as follows:

1. Exhaust the supply (factory capacity) of each row before moving down to the next row.
2. Exhaust the demand (warehouse) requirements of each column before moving to the next column to the right.
3. Check that all supply and demand requirements are met.

Initial Solution Using the Northwest Corner Rule

It takes five steps in this example to make the initial shipping assignments.

1. Beginning in the upper left-hand corner, assign 100 units from Des Moines to Albuquerque.
   - This exhausts the capacity or supply at the Des Moines factory.
   - But it still leaves the warehouse at Albuquerque 200 desks short.
   - Next, move down to the second row in the same column.
2. Assign 200 units from Evansville to Albuquerque.
   - This meets Albuquerque's demand for a total of 300 desks.
   - The Evansville factory has 100 units remaining, so we move to the right to the next column of the second row.

Initial Solution Using the Northwest Corner Rule

Steps 3 and 4 in this example are to make the initial shipping assignments.

3. Assign 100 units from Evansville to Boston.
   - The Evansville supply has now been exhausted, but Boston's warehouse is still short by 100 desks.
   - At this point, move down vertically in the Boston column to the next row.
4. Assign 100 units from Fort Lauderdale to Boston.
   - This shipment will fulfill Boston's demand for a total of 200 units.
   - Note that the Fort Lauderdale factory still has 200 units available that have not been shipped.

Initial Solution Using the Northwest Corner Rule

Final step for the initial shipping assignments.

5. Assign 200 units from Fort Lauderdale to Cleveland.
   - This final move exhausts Cleveland's demand *and* Fort Lauderdale's supply.
   - This always happens with a balanced problem.
   - The initial shipment schedule is now complete and shown in the next slide.

(Continued: next slide)

Initial Solution North West Corner Rule

The Executive Furniture Corporation

	Albuquerque (A)	Boston (B)	Cleveland (C)	Factory Capacity
Des Moines (D)	100 (5)	(4)	(3)	100
Evansville (E)	200 (8)	100 (4)	(3)	300
Fort Lauderdale (F)	(9)	100 (7)	200 (5)	300
Warehouse Req.	300	200	200	700

Initial Solution Using the Northwest Corner Rule

- This solution is feasible since demand and supply constraints are all satisfied.
  - It must be evaluated to see if it is optimal.
  - Compute an improvement index for each empty cell using either the stepping-stone method or the MODI method.
  - If this indicates a better solution is possible, use the stepping-stone path to move from this solution to improved solutions until an optimal solution is found.

The Five Steps of the Stepping-Stone Method

1. Select any unused square to evaluate.
2. Begin at this square. Trace a closed path back to the original square via squares that are currently being used (only horizontal or vertical moves allowed).
3. Beginning with a plus (+) sign at the unused square, place alternate minus (-) signs and plus signs on each corner square of the closed path just traced.
4. Calculate an *improvement index* by adding together the unit cost figures found in each square containing a plus sign and then subtracting the unit costs in each square containing a minus sign.

The Five Steps of the Stepping-Stone Method

(Continued)

5. Repeat steps 1 to 4 until an improvement index has been calculated for all unused squares.
   - If all indices computed are greater than or equal to zero, an optimal solution has been reached.
   - If not, it is possible to improve the current solution and decrease total shipping costs.

- The next several slides show the results of following the preceding 5 steps.

Stepping-Stone Method - The Des Moines-to-Boston Route

The Executive Furniture Corporation

	Albuquerque (A)	Boston (B)	Cleveland (C)	Factory Capacity
Des Moines (D)	100 (5) −	Start (4) +	(3)	100
Evansville (E)	200 (8) +	100 (4) −	(3)	300
Fort Lauderdale (F)	(9)	100 (7)	200 (5)	300
Warehouse Req.	300	200	200	700

Stepping-Stone Method - The Des Moines-to-Boston Route

Improvement index =

+4 – 5 + 8 – 4 = +3

Stepping-Stone Method - The Ft. Lauderdale-to-Albuquerque Route

The Executive Furniture Corporation

	Albuquerque (A)	Boston (B)	Cleveland (C)	Factory Capacity
Des Moines (D)	100 [5]	[4]	[3]	100
Evansville (E)	200 [8] -	100 [4] +	[3]	300
Fort Lauderdale (F)	Start [9] +	100 [7] -	200 [5]	300
Warehouse Req.	300	200	200	700

Stepping-Stone Method - The Ft. Lauderdale-to-Albuquerque Route

Improvement index =

+4 – 8 + 9 – 7 = -2

Stepping-Stone Method - The Evansville-to-Cleveland Route

The Executive Furniture Corporation

	Albuquerque (A)	Boston (B)	Cleveland (C)	Factory Capacity
Des Moines (D)	100 [5]	[4]	[3]	100
Evansville (E)	200 [8]	100 [4] -	Start [3] +	300
Fort Lauderdale (F)	[9]	100 [7] +	200 [5] -	300
Warehouse Req.	300	200	200	700

Stepping-Stone Method - The Evansville-to-Cleveland Route

Improvement index =

+3 – 4 + 7 – 5 = +1

Stepping-Stone Method - The Des Moines-to-Cleveland Route

The Executive Furniture Corporation

	Albuquerque (A)	Boston (B)	Cleveland (C)	Factory Capacity
Des Moines (D)	100 [5] -	[4]	Start [3] +	100
Evansville (E)	200 [8] +	100 [4] -	[3]	300
Fort Lauderdale (F)	[9]	100 [7] +	200 [5] -	300
Warehouse Req.	300	200	200	700

Stepping-Stone Method - The Des Moines-to-Cleveland Route

Improvement index =

+3 – 5 + 8 – 4 + 7 - 5 = +4

Selecting the Cell for Improvement

- **The cell with the best negative improvement index is selected. This cell will be filled with as many units as possible.**
- **In this example, the only cell with a negative improvement index is FA (Ft. Lauderdale to Albuquerque)**

Stepping-Stone Method - The Ft. Lauderdale-to-Albuquerque Route

The Executive Furniture Corporation

	Albuquerque (A)	Boston (B)	Cleveland (C)	Factory Capacity
Des Moines (D)	100 [5]	[4]	[3]	100
Evansville (E)	- 200 [8]	+ 100 [4]	[3]	300
Fort Lauderdale (F)	Start + [9]	- 100 [7]	200 [5]	300
Warehouse Req.	300	200	200	700

How Many Units Are Added?

- **If cell FA is to be filled, whatever is added to this is subtracted from EA and FB. Since FB only has 100 units, this is all that can be added to FA.**

Stepping-Stone Method: An Improved Solution

The Executive Furniture Corporation

	Albuquerque (A)	Boston (B)	Cleveland (C)	Factory Capacity
Des Moines (D)	100 [5]	[4]	[3]	100
Evansville (E)	100 [8]	200 [4]	[3]	300
Fort Lauderdale (F)	100 [9]	[7]	200 [5]	300
Warehouse Req.	300	200	200	700

Continuing the Process

- **All empty cells are now evaluated again. If any cell has a negative index, the process continues and a new solution is found.**

Stepping-Stone Method: Improvement Indices

The Executive Furniture Corporation

	Albuquerque (A)	Boston (B)	Cleveland (C)	Factory Capacity
Des Moines (D)	100 [5]	+3 [4]	+2 [3]	100
Evansville (E)	100 [8]	200 [4]	-1 [3]	300
Fort Lauderdale (F)	100 [9]	+2 [7]	200 [5]	300
Warehouse Req.	300	200	200	700

Third and Final Solution

The Executive Furniture Corporation

	Albuquerque (A)	Boston (B)	Cleveland (C)	Factory Capacity
Des Moines (D)	100 [5]	[4]	[3]	100
Evansville (E)	[8]	200 [4]	100 [3]	300
Fort Lauderdale (F)	200 [9]	[7]	100 [5]	300
Warehouse Req.	300	200	200	700

The MODI Method

- The MODI (*modified distribution*) method allows improvement indices quickly to be computed for each unused square without drawing all of the closed paths.
- Because of this, it can often provide considerable time savings over the stepping-stone method for solving transportation problems.
- In applying the MODI method, begin with an initial solution obtained by using the northwest corner rule.

The MODI Method

- But now must compute a value for each row (call the values $R1$, $R2$, $R3$ if there are three rows) and for each column ($K1$, $K2$, $K3$) in the transportation table.
- The next slide summarizes the five steps in the MODI Method.

MODI Method: Five Steps

1. Compute the values for each row and column: set $R_i + K_j = C_{ij}$ for those squares *currently used or occupied*.
2. Set $R_1 = 0$.
3. Solve the system of equations for R_i and K_j values.
4. Compute the improvement index for each unused square by the formula: *Improvement Index* = $C_{ij} - R_i - K_j$
5. Select the best negative index and proceed to solve the problem as you did using the stepping-stone method.

Vogel's Approximation Alternative to the Northwest Corner Method

- VAM is not as simple as the northwest corner method, but it provides a very good initial solution, usually one that is the *optimal* solution.
- VAM tackles the problem of finding a good initial solution by taking into account the costs associated with each route alternative.
  - This is something that the northwest corner rule does not do.
- To apply VAM, we first compute for each row and column the penalty faced if we should ship over the *second best* route instead of the *least-cost* route.

The Six Steps for Vogel's Approximation

1. For each row/column, find difference between two lowest costs.
   - Opportunity cost
2. Select the greatest opportunity cost.
3. Assign as many units as possible to lowest cost square in row/column with greatest opportunity cost.
4. Eliminate row or column that has been completely satisfied.
5. Recompute the opportunity costs for remaining rows/columns.

Special Problems in Transportation Method

- Unbalanced problem
  - Demand less than supply
  - Demand greater than supply
- Degeneracy
- More than one optimal solution

Unbalanced Transportation Problems

- In real-life problems, total demand is not equal to total supply.
- These *unbalanced problems* can be handled easily by using *dummy sources* or *dummy destinations*.
- If total supply is greater than total demand, a dummy destination (warehouse), with demand exactly equal to the surplus, is created.
- If total demand is greater than total supply, introduce a dummy source (factory) with a supply equal to the excess of demand over supply.

Unbalanced Transportation Problems

- Regardless of whether demand or supply exceeds the other, shipping cost coefficients of zero are assigned to each dummy location or route because no shipments will actually be made from a dummy factory or to a dummy warehouse.
- Any units assigned to a dummy destination represent excess capacity, and units assigned to a dummy source represent unmet demand.

Unbalanced Problem Demand Less than Supply

- Suppose that the Des Moines factory increases its rate of production to 250 desks.
  - That factory's capacity used to be 100 desks per production period.
- The firm is now able to supply a total of 850 desks each period.
- Warehouse requirements remain the same so the row and column totals do not balance.

Unbalanced Problem Demand Less than Supply

(Continued)

- To balance this type of problem, simply add a dummy column that will represent a fake warehouse requiring 150 desks.
  - This is somewhat analogous to adding a slack variable in solving an LP problem.
- Just as slack variables were assigned a value of zero dollars in the LP objective function, the shipping costs to this dummy warehouse are all set equal to zero.

Unbalanced Problem Demand Less than Supply

The Executive Furniture Corporation

	A	B	C	Dummy	
D					**250**
E					**300**
F					**300**
	300	**200**	**200**	**150**	**850**

A cost of "0" is given to all the cells in the dummy column.

Example - Demand Less than Supply

	Customer 1	Customer 2	Dummy	Factory Capacity
Factory 1	8	5	0	170
Factory 2	15	10	0	130
Factory 3	3	9	0	80
Customer Requirements	150	80	150	380

Unbalanced Problem Supply Less than Demand

- The second type of unbalanced condition occurs when total demand is greater than total supply.
- This means that customers or warehouses require more of a product than the firm's factories can provide.
- In this case we need to add a dummy row representing a fake factory.
- The new factory will have a supply exactly equal to the difference between total demand and total real supply.
- The shipping costs from the dummy factory to each destination will be zero.

Example - Supply Less than Demand

	Customer 1	Customer 2	Customer 3	Factory Capacity
Factory 1	8	5	16	170
Factory 2	15	10	7	130
Dummy	0	0	0	80
Customer Requirements	150	80	150	380

Degeneracy

- Degeneracy occurs when the number of occupied squares or routes in a transportation table solution is less than the number of rows plus the number of columns minus 1.
  - # Occupied Squares = Rows + Columns – 1
- Such a situation may arise in the initial solution or in any subsequent solution.
  - Degeneracy requires a special procedure to correct the problem.
- Without enough occupied squares to trace a closed path for each unused route, it would be impossible to apply the *stepping-stone method* or to calculate the *R* and *K* values needed for the *MODI technique*.

Degeneracy

- To handle degenerate problems, create an artificially occupied cell.
- * That is, place a zero (representing a fake shipment) in one of the unused squares and then treat that square as if it were occupied.
- The square chosen must be in such a position as to allow *all* stepping-stone paths to be closed.
- * Although there is usually a good deal of flexibility in selecting the unused square that will receive the zero.

More Than One Optimal Solution

- **As with LP problems, it is possible for a Transportation Problem to have multiple optimal solutions.**
- **Such is the case when one or more of the improvement indices that we calculate for each unused square is zero in the optimal solution.**
  - This means that it is possible to design alternative shipping routes with the same total shipping cost.
- **The alternate optimal solution can be found by shipping the most to this unused square (with index = 0) using a stepping-stone path.**
- **Practically speaking, multiple optimal solutions provide management with greater flexibility in selecting and using resources.**

Maximization Transportation Problems

- If the objective in a transportation problem is to maximize profit, a minor change is required in the transportation algorithm.
- Since the improvement index for an empty cell indicates how the objective function value will change if one unit is placed in that empty cell,
  - the optimal solution is reached when all the improvement indices are negative or zero.
- If any index is positive, the cell with the largest positive improvement index is selected to be filled using a stepping-stone path.
- This new solution is evaluated and the process continues until there are no positive improvement indices.

Unacceptable Or Prohibited Routes

- At times there are transportation problems in which one of the sources is unable to ship to one or more of the destinations.
  - When this occurs, the problem is said to have an *unacceptable* or *prohibited route.*
- In a minimization problem, such a prohibited route is assigned a very high cost to prevent this route from ever being used in the optimal solution.
- After this high cost is placed in the transportation table, the problem is solved using the techniques previously discussed.
- In a maximization problem, the very high cost used in minimization problems is given a negative sign, turning it into a very bad profit.

The Assignment Model

- The second special-purpose LP algorithm is the *assignment method.*
- Each assignment problem has associated with it a table, or matrix.
- Generally, the rows contain the objects or people we wish to assign, and the columns comprise the tasks or things we want them assigned to.
- The numbers in the table are the costs associated with each particular assignment.

The Assignment Model

- An assignment problem can be viewed as a transportation problem in which
  - the capacity from each source (or person to be assigned) is 1 *and*
  - the demand at each destination (or job to be done) is 1.
- Such a formulation could be solved using the transportation algorithm, but it would have a severe degeneracy problem.
- However, this type of problem is very easy to solve using the assignment method.

Assignment Problem Example

	Project		
Person	1	2	3
Adams	$11	$14	$6
Brown	$8	$10	$11
Cooper	$9	$12	$7

The Steps of the Assignment Method

1. Find the opportunity cost table by:
   a) Subtracting the smallest number in each row of the original cost table or matrix from every number in that row.
   b) Then subtracting the smallest number in each column of the table obtained in part (a) from every number in that column.

Assignment Problem Example

	Project		
Person	1	2	3
Adams	$11	$14	$6
Brown	$8	$10	$11
Cooper	$9	$12	$7

	Project		
Person	1	2	3
Adams	11-6	14-6	6-6
Brown	8-8	10-8	11-8
Cooper	9-7	12-7	7-7

Assignment Problem Example

	Project		
Person	1	2	3
Adams	5	8	0
Brown	0	2	3
Cooper	2	5	0

Subtract smallest number in each column. Note columns with 0s do not change.

	Project		
Person	1	2	3
Adams	5	8-2	0
Brown	0	2-2	3
Cooper	2	5-2	0

Assignment Problem Example

	Project		
Person	1	2	3
Adams	5	6	0
Brown	0	0	3
Cooper	2	3	0

Steps of the Assignment Method

(continued)

2. Test the table resulting from step 1 to see whether an optimal assignment can be made.
   - The procedure is to draw the minimum number of vertical and horizontal straight lines necessary to cover all zeros in the table.
   - If the number of lines equals either the number of rows or columns in the table, an optimal assignment can be made.
   - If the number of lines is less than the number of rows or columns, then proceed to step 3.

Assignment Problem Example

	Project		
Person	1	2	3
Adams	~~5~~	~~6~~	~~0~~
Brown	0	0	3
Cooper	2	3	0

Steps of the Assignment Method

(continued)

3. Revise the present opportunity cost table.
   - This is done by subtracting the smallest number not covered by a line from every other uncovered number.
   - This same smallest number is also added to any number(s) lying at the intersection of horizontal and vertical lines.
   - We then return to step 2 and continue the cycle until an optimal assignment is possible.

Assignment Problem Example

	Project		
Person	1	2	3
Adams	5-2	6-2	0 +2
Brown	0	0	3
Cooper	2-2	3-2	0

	Project		
Person	1	2	3
Adams	3	4	0
Brown	0	0	5
Cooper	0	1	0

Unbalanced Assignment Problems

- Often the number of people or objects to be assigned does not equal the number of tasks or clients or machines listed in the columns, and the problem is *unbalanced*.
  - When this occurs, and there are more rows than columns, simply add a *dummy column* or task (similar to how unbalanced transportation problems were dealt with earlier).

	Job		
Person	1	2	Dummy
Smith	21	26	0
Jones	20	21	0
Garcia	22	20	0

Unbalanced Assignment Problems

(continued)

- If the number of tasks that need to be done exceeds the number of people available, add a *dummy row*.
  - This creates a table of equal dimensions and allows us to solve the problem as before.
- Since the dummy task or person is really nonexistent, it is reasonable to enter zeros in its row or column as the cost or time estimate.

	Task		
Person	1	2	3
McCormack	135	165	88
Perdue	145	162	86
Dummy	0	0	0

Maximization Assignment Problems

- Some assignment problems are phrased in terms of maximizing the payoff, profit, or effectiveness of an assignment instead of minimizing costs.
- It is easy to obtain an equivalent minimization problem by converting all numbers in the table to opportunity costs.
  - This is brought about by subtracting every number in the original payoff table from the largest single number in that table.

Maximization Assignment Problems

- The transformed entries represent opportunity costs:
  - it turns out that minimizing opportunity costs produces the same assignment as the original maximization problem.
- Once the optimal assignment for this transformed problem has been computed, the total payoff or profit is found by adding the original payoffs of those cells that are in the optimal assignment.