

# **Linear Programming: The Simplex Method**

## **Learning Objectives**

### **Students will be able to:**

1. Convert LP constraints to equalities with slack, surplus, and artificial variables.
2. Set up and solve LP problems with simplex tableaus.
3. Interpret the meaning of every number in a simplex tableau.
4. Recognize special cases such as infeasibility, unboundedness, and degeneracy.
5. Use the simplex tables to conduct sensitivity analysis.
6. Construct the dual problem from the primal problem.

## **Introduction**

- Graphical methods are fine for 2 variables.
- But most LP problems are too complex for simple graphical procedures.
- The Simplex Method:
  - o examines corner points, like in graphing;
  - o systematically examines corner points, using algebra, until an optimal solution is found;
  - o does its searching iteratively.

## **Introduction**

### *Why study the Simplex Method?*

- The Simplex Method:
  - o Provides the optimal solution to the  $X_i$  variables and the maximum profit (or minimum cost).
  - o Provides important economic information.
- Understanding how the Simplex Method works is important because
  - o it allows the use of computer generated solutions to be run successfully, and
  - o it allows for understanding how to interpret LP computer printouts.

## Setting Up the Simplex Tableau

- The first example examined is to solve a maximization problem.
- The Flair Furniture Company from Chapter 7 is examined.
- First, start by reviewing the original equations.

## Flair Furniture Company

*Hours Required to Produce One Unit*

Department	T Tables	C Chairs	Available Hours This Week
Painting/Varnishing	2	1	100
Carpentry	4	3	240
Profit/unit	\$7	\$5	

Objective: *Maximize:*  $7T + 5C$

Constraints:

$$2T + 1C \leq 100 \text{ (painting \& varnishing)}$$

$$4T + 3C \leq 240 \text{ (carpentry)}$$

$$T, C \geq 0 \text{ (nonnegativity constraints)}$$

## Making Constraints into Equations

### *1<sup>st</sup> Step*

- Convert each inequality constraint into an equation.
- Less-than-or-equal-to constraints ( $\leq$ ) are converted to equations by adding a *slack variable* to each.
  - Slack variables represent unused resources.
- For the Flair Furniture problem, define the slacks as:
  - $S_1$  = unused hours in the painting dept
  - $S_2$  = unused hours in the carpentry dept
- The constraints are now written as:
  - $2T + 1C + S_1 = 100$
  - $4T + 3C + S_2 = 240$

## Making Constraints into Equations

### *1<sup>st</sup> Step, continued*

- Slack variables not appearing in an equation are added with a coefficient of 0.
  - This allows all the variables to be monitored at all times.
- The final Simplex Method equations appear as:
  - $2T + 1C + 1S_1 + 0S_2 = 100$
  - $4T + 3C + 0S_1 + 1S_2 = 240$
  - $T, C, S_1, S_2 \geq 0$
- The slacks are added to the objective coefficient with 0 profit coefficients.
- The objective function, then, is:
  - maximize profit:  
 $\$7T + \$5C + \$0S_1 + \$0S_2$

## An Algebraic Initial Solution

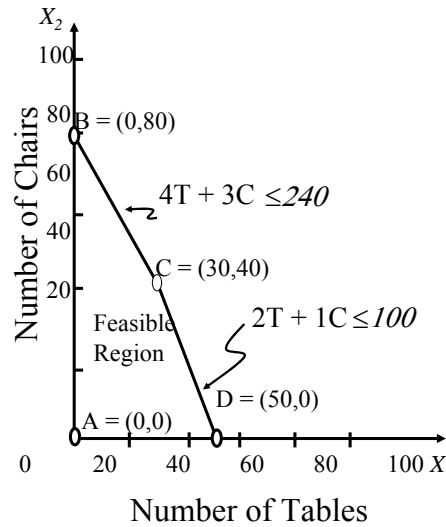
- There are now 2 ( $n$ ) equations and 4 variables.
- When the number of variables is  $>$  than the number of equations, the system can only be solved by setting some of the variables equal to 0.
- A basic feasible solution to a system of  $n$  equations is found by setting all but  $n$  variables equal to 0 and solving for the other variables.
- Solutions found in this manner are called *basic feasible solutions*.

## An Algebraic Initial Solution

Using the figure on the next slide, the following information can be observed:

- The Simplex Method starts with an initial feasible solution with all real variables (T and C) set to 0 [Point A on the graph].
- The Simplex Method will always start at this point and then move up or over to the corner point that provides the most improved profit [Points B or D].
- The method will move to a new corner point [C], which is the optimal point in this example.
- The method considers only feasible solutions and will only touch the corner points of the feasible region.

## Flair Furniture Company's Feasible Region & Corner Points



## Flair Furniture's First Simplex Tableau

*The Next Step*

- All the coefficients of all the equations and objective function need to be tabular form.
- The two constraints are written below.
  - o The numbers (2,1,1,0) in the first row are the coefficients of the first equation,
  - o The numbers (4,3,0,1) in the second row are the coefficients of the second equation.
- The first *simplex tableau* is on the next slide.

Constraints in tabular form:

<i>Solution Mix</i>	T	C	$S_1$	$S_2$	<i>Quantity</i>
$S_1$	2	1	1	0	100
$S_2$	4	3	0	1	240

*Constraint equation rows*

## Flair Furniture's Initial Simplex Tableau

$C_j$							
	<i>Solution Mix</i>		<i>Real Variables Columns</i>		<i>Slack Variables Columns</i>		
		T	C	$S_1$	$S_2$	Quantity	
\$7							<i>Profit per unit row</i>
\$5							
\$0							
\$0	$S_1$	2	1	1	0	100	<i>Constraint equation rows</i>
\$0	$S_2$	4	3	0	1	240	
	$Z_j$	\$0	\$0	\$0	\$0	\$0	<i>Gross Profit row</i>
	$C_j - Z_j$	\$7	\$5	\$0	\$0	\$0	<i>Net Profit row</i>

## Flair Furniture's Initial Simplex Tableau

$C_j$							
	<i>Solution Mix</i>		<i>Real Variables Columns</i>		<i>Slack Variables Columns</i>		
		T	C	$S_1$	$S_2$	Quantity	
\$7							<i>Profit per unit row</i>
\$5							
\$0							
\$0	$S_1$	2	1	1	0	100	<i>Constraint equation rows</i>
\$0	$S_2$	4	3	0	1	240	
	$Z_j$	\$0	\$0	\$0	\$0	\$0	<i>Gross Profit row</i>
	$C_j - Z_j$	\$7	\$5	\$0	\$0	\$0	<i>Net Profit row</i>

## Flair Furniture's First Simplex Tableau

- The initial solution is called a *basic feasible solution* and can be written as a vector:

$$\begin{bmatrix} T \\ C \\ S_1 \\ S_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 100 \\ 240 \end{bmatrix}$$

- The solution mix is referred to as the *basis* and all variables in the basis are called *basic*.
- *Nonbasic variables* are those set equal to zero in the basis.

## Flair Furniture's First Simplex Tableau

### *Substitution Rates*

- Substitution rates are numbers in the body of the simplex table.
- The entries in the table in the columns under each variable are the coefficients of that variable.

<i>Solution Mix</i>	T	C	$S_1$	$S_2$	<i>Quantity</i>
$S_1$	2	1	1	0	100
$S_2$	4	3	0	1	240

Under T are the coefficients  $\begin{pmatrix} 2 \\ 4 \end{pmatrix}$

Under C are the coefficients  $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$

Under  $S_1$  are the coefficients  $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

Under  $S_2$  are the coefficients  $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$



## Flair Furniture's First Simplex Tableau

### *Substitution Rates, continued*

- The numbers in the body of the simplex tableau can be thought of as substitution rates.
- *For example,*
  - To make T larger than 0, every unit increase of T will require 2 units of  $S_1$  and 4 units of  $S_2$  must be removed.
  - The substitution rates for each unit of C is 1 unit of  $S_1$  and 3 units of  $S_2$ .
- For any variable ever to appear in the solution mix column, it must have the number 1 someplace in its column and 0's in every other place in that column.

Column  $S_1$  has  $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ , so it is in the solution.

Column  $S_2$  has  $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ , so it is in the solution.

## Flair Furniture's First Simplex Tableau

### *Adding the Objective Function*

- Next step is to add a row to reflect the objective function values for each variable.
- These values are called contribution rates and are labeled as  $C_j$ . They appear above each variable, as in the following tableau:

$C_j$	$\rightarrow$	\$7	\$5	\$0	\$0	
$\downarrow$	<i>Solution Mix</i>	T	C	$S_1$	$S_2$	<i>Quantity</i>
\$0	$S_1$	2	1	1	0	100
\$0	$S_2$	4	3	0	1	240

- The unit profit rates are found in the top row and the leftmost column.
  - $C_j$  indicates the unit profit for each variable currently in the solution mix.

## Flair Furniture's First Simplex Tableau

### *The $Z_j$ and $C_j - Z_j$ Rows*

- The  $Z_j$  value for the *Quantity* column provides the total contribution (gross profit in this case) of the given solution.
- The  $Z_j$  values for the other columns (variables  $T$ ,  $C$ ,  $S_1$ , and  $S_2$ ) represent the gross profit *given up* by adding one unit of this variable into the current solution.
- In this example, there is no profit *lost* by adding one unit of either  $T$  (tables),  $C$  (chairs),  $S_1$ , or  $S_2$ .

## Flair Furniture's First Simplex Tableau

### *The $Z_j$ and $C_j - Z_j$ Rows*

- The  $C_j - Z_j$  number in each column represents the net profit that will result from introducing 1 unit of each product or variable into the solution,
  - o i.e., the profit gained minus the profit given up.
  - o It is not calculated for the quantity column.
- To compute these numbers, simply subtract the  $Z_j$  total for each column from the  $C_j$  value at the very top of that variable's column.
- The calculations for the net profit per unit (the  $C_j - Z_j$  row) in this example follow.

## Flair Furniture's First Simplex Tableau

*The  $Z_j$  and  $C_j - Z_j$  Rows*

		COLUMN			
calculations for the net profit per unit		T	C	$S_1$	$S_2$
	$C_j$	\$7	\$5	\$0	\$0
	$Z_j$	<u>\$0</u>	<u>\$0</u>	<u>\$0</u>	<u>\$0</u>
	$C_j - Z_j$	\$7	\$5	\$0	\$0

- When a profit of \$0 is computed, the initial solution is not optimal.
- Examining the numbers in the  $C_j - Z_j$  row shows that the total profit can be increased by \$7 for each unit of  $T$  (tables) and by \$5 for each unit of  $C$  (chairs) added to the solution mix.
  - A negative number in the  $C_j - Z_j$  row would tell us that profits would *decrease* if the corresponding variable were added to the solution mix.
- An optimal solution is reached in the simplex method when the  $C_j - Z_j$  row contains no positive numbers.

## Simplex Steps for Maximization

- After the initial tableau is completed, proceed through a series of five steps to compute all the numbers needed in the next tableau.
  - The calculations are not difficult, but they are complex; the smallest arithmetic error can produce a wrong answer.

*The 5 Steps*

1. Choose the variable with the greatest positive  $C_j - Z_j$  to enter the solution.
2. Determine the row to be replaced by selecting that one with the smallest (non-negative) quantity-to-pivot-column ratio.
3. Calculate the new values for the pivot row.
4. Calculate the new values for the other row(s).
5. Calculate the  $C_j$  and  $C_j - Z_j$  values for this tableau.
  - If there are any  $C_j - Z_j$  values greater than zero, return to step 1.

### Pivot Row, Pivot Number Identified in the Initial Simplex Tableau

$C_j$ →		\$7	\$5	\$0	\$0	
↓	<i>Solution Mix</i>					
		<i>T</i>	<i>C</i>	<i>S<sub>1</sub></i>	<i>S<sub>2</sub></i>	<i>Quantity</i>
\$0	<i>S<sub>1</sub></i>	2	1	1	0	100 ← <i>Pivot row</i>
\$0	<i>S<sub>2</sub></i>	4	3	0	1	240
		<i>Pivot number</i>				
	$Z_j$	\$0	\$0	\$0	\$0	\$0
	$C_j - Z_j$	\$7	\$5	\$0	\$0	
		<i>Pivot column</i>				
		<i>Largest (<math>C_j - Z_j</math>) value</i>				

### Pivot Row Changed

$C_j$ →		\$7	\$5	\$0	\$0		Row divided by 2
↓	<i>Solution Mix</i>						
		<i>T</i>	<i>C</i>	<i>S<sub>1</sub></i>	<i>S<sub>2</sub></i>	<i>Quantity</i>	
\$7	<i>T</i>	1	1/2	1/2	0	50	↙
\$0	<i>S<sub>2</sub></i>	4	3	0	1	240	↙
	$Z_j$						
	$C_j - Z_j$						

This row will be changed next to get a zero in the pivot column.

## Calculating the New $S_2$ Row for Flair's Second Tableau

Equation 9-1

$$\begin{pmatrix} \text{New} \\ \text{Row} \\ \text{Numbers} \end{pmatrix} = \begin{pmatrix} \text{Numbers} \\ \text{in old} \\ \text{row} \end{pmatrix} - \left[ \begin{pmatrix} \text{Number} \\ \text{above} \\ \text{or below} \\ \text{pivot} \\ \text{number} \end{pmatrix} \times \begin{pmatrix} \text{Corresponding} \\ \text{number} \\ \text{in the} \\ \text{new row} \end{pmatrix} \right]$$

$$\begin{array}{rclclcl} 0 & = & 4 & - & (4) & \times & (1) \\ 1 & = & 3 & - & (4) & \times & (1/2) \\ -2 & = & 0 & - & (4) & \times & (1/2) \\ 1 & = & 1 & - & (4) & \times & (0) \\ 40 & = & 240 & - & (4) & \times & (50) \end{array}$$


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## Completed Second Simplex Tableau for Flair Furniture

$C_j \longrightarrow$		\$7	\$5	\$0	\$0	Row divided by 2
↓	<i>Solution Mix</i>	T	C	$S_1$	$S_2$	Quantity
\$7	T	1	1/2	1/2	0	50
\$0	$S_2$	0	1	-2	1	40
	$Z_j$	\$7	\$7/2	\$7/2	\$0	\$350
	$C_j - Z_j$	\$0	\$3/2	-\$7/2	\$0	

Row calculated using Equation 9-1 as shown in previous slide

### Pivot Row, Column, and Number Identified in Second Simplex Tableau

$C_j$ →			\$7	\$5	\$0	\$0	
↓	<i>Solution Mix</i>						
		<i>T</i>	<i>C</i>	$S_1$	$S_2$	<i>Quantity</i>	
\$7	<i>T</i>	1	1/2	1/2	0	50	
\$0	$S_2$	0	(1)	-2	1	40	← <i>Pivot row</i>
				← <i>Pivot number</i>			
	$Z_j$	\$7	\$7/2	\$7/2	\$0	\$350	
						(Total Profit)	
	$C_j - Z_j$	\$0	\$3/2	-\$7/2	\$0		
			↑	← <i>Pivot column</i>			

### Calculating the New *T* Row for Flair's Third Tableau

$$\begin{pmatrix} \text{New Row} \\ \text{Numbers} \end{pmatrix} = \begin{pmatrix} \text{Numbers} \\ \text{in old row} \end{pmatrix} - \left[ \begin{pmatrix} \text{Number above or below pivot number} \end{pmatrix} \times \begin{pmatrix} \text{Corresponding number in the new row} \end{pmatrix} \right]$$

1	=	1	-	(1/2)	x	(0)
0	=	1/2	-	(1/2)	x	(1)
3/2	=	1/2	-	(1/2)	x	(-2)
-1/2	=	0	-	(1/2)	x	(1)
30	=	50	-	(1/2)	x	(40)

## Final Simplex Tableau for the Flair Furniture Problem

$C_j$ →		\$7	\$5	\$0	\$0	
↓	<i>Solution</i>					
	<i>Mix</i>	<i>T</i>	<i>C</i>	<i>S<sub>1</sub></i>	<i>S<sub>2</sub></i>	<i>Quantity</i>
\$7	<i>T</i>	1	0	3/2	-1/2	30
\$5	<i>C</i>	0	1	-2	1	40
	$Z_j$	\$7	5	\$1/2	\$3/2	\$410
	$C_j - Z_j$	\$0	\$0	-\$1/2	-\$3/2	

Since every number in the last row is 0 or negative, an optimal solution has been found. The solution is:

$T = 30$  tables                       $S_1 = 0$  slack hours in painting  
 $C = 40$  chairs                       $S_2 = 0$  slack hours in carpentry  
 profit = \$410 for the optimal solution

## Surplus and Artificial Variables

- To use the simplex method with greater-than-or-equal-to ( $\geq$ ) constraints and equalities, each of these must be converted to a special form similar to that made for the less-than-or-equal-to ( $\leq$ ) constraints.
- If they are not, the simplex technique is unable to set up an initial solution in the first tableau.
- Consider the following two constraints to see how to convert typical constraints greater-than-or-equal-to ( $\geq$ ) constraints and equalities:

Constraint 1:  $5X_1 + 10X_2 + 8X_3 \geq 210$

Constraint 2:  $25X_1 + 30X_2 = 900$

## Surplus and Artificial Variables

- Greater-than-or-equal-to ( $\geq$ ) constraints (e.g., constraint #1) require a different approach than do the less-than-or-equal-to ( $\leq$ ) constraints seen in the Flair Furniture problem.
- They involve the subtraction of a *surplus variable* rather than the addition of a slack variable.
- The surplus variable tells us how much the solution exceeds the constraint resource.
- Because of its analogy to a slack variable, surplus is sometimes simply called *negative slack*.
- To convert the first constraint, begin by subtracting a surplus variable,  $S_1$ , to create an equality.

## Surplus and Artificial Variables

- There is one more step in preparing a  $\geq$  constraint for the simplex method.
- Artificial variables are needed in  $\geq$  and = constraints.
- Simply add the artificial variable,  $A_1$ , to the constraint as follows:  
**Constraint 1 completed:**  
$$5X_1 + 10X_2 + 8X_3 - S_1 + A_1 \geq 210$$
- Now, not only the  $X_1$ ,  $X_2$ , and  $X_3$  variables may be set to 0 in the initial simplex solution, but the  $S_1$  surplus variable as well.
- This leaves  $A_1 = 210$ .



## Surplus and Artificial Variables

- Constraint 2 is already an equality.

*So why worry about it?*

- To be included in the initial simplex solution even an equality must have an artificial variable added to it.

**Constraint 2** rewritten:

$$25X_1 + 30X_2 + A_2 = 900$$

- The reason for artificial variables in an equality constraint concerns finding an initial LP solution.
- Depending on the complexity of the constraint, it can be *extremely* difficult to visualize a set of initial solutions.
- By adding artificial variables, such as  $A_2$ , an automatic initial solution can be found.
- In this case, when  $X_1$  and  $X_2$  are set equal to 0,  $A_2 = 900$ .

## Surplus and Artificial Variables

- Artificial variables have no meaning in a physical sense and are nothing more than computational tools for generating initial LP solutions.
- Before the final simplex solution has been reached, all artificial variables must be gone from the solution mix.
- Artificial variables have no physical meaning and drop out of the solution mix before the final tableau.
- This matter is handled through the problem's objective function.

## Surplus and Artificial Variables

- Whenever an artificial or surplus variable is added to one of the constraints, it must also be included
  - o in the other equations and
  - o in the problem's objective function.just as was done for slack variables.
- Since artificial variables must be forced out of the solution, assign a very high cost to each.
- Simply use the letter ***M*** to represent a very large number.
- Surplus variables, like slack variables, carry a zero cost.
- In maximization problems, use negative ***M***.

## Surplus and Artificial Variables

- If a problem had an objective function that read:
  - minimize cost =  $\$5X_1 + \$9X_2 + \$7X_3$
- and constraints such as the two mentioned previously, the completed objective function and constraints would appear as follows:

minimize cost =  
 $\$5X_1 + \$9X_2 + \$7X_3 + \$0S_1 + \$MA_1 + \$MA_2$

subject to:

$$5X_1 + 10X_2 + 8X_3 - 1S_1 + 1A_1 + 0A_2 = 210$$
$$25X_1 + 30X_2 + 0X_3 + 0S_1 + 0A_1 + 1A_2 = 900$$

## Procedures LP Minimization

- I. Formulate the LP problem's objective function and constraints.
- II. Include:
  - \* slack variables in each less-than-or-equal-to constraint,
  - \* artificial variables in each equality constraint, and
  - \* both surplus and artificial variables in each greater-than-or-equal-to constraint.Then add all of these variables to the problem's objective function.
- III. Develop an initial simplex tableau with artificial and slack variables in the basis and their variables (the  $X_i$ 's) set equal to 0. Compute the  $Z_j$  and  $C_j - Z_j$  values for this tableau.

## Simplex Steps for Minimization

- IV. Follow these five steps until an optimal solution has been reached:
  1. Choose the variable with the greatest negative  $C_j - Z_j$  to enter the solution.
  2. Determine the row to be replaced by selecting that one with the smallest (non-negative) quantity-to-pivot-column ratio.
  3. Calculate the new values for the pivot row.
  4. Calculate the new values for the other row(s).
  5. Calculate the  $C_j$  and  $C_j - Z_j$  values for this tableau. If there are any  $C_j - Z_j$  values less than zero, return to step 1.

## Special Cases

- In Chapter 7 some special cases that may arise when solving LP problems graphically were addressed (Section 8 of Chapter 7).
- These cases are discussed again, this time as they refer to the simplex method.
- These four cases are:
  1. Infeasibility
  2. Unbounded Solutions
  3. Degeneracy
  4. Multiple Optimal Solutions

## Special Cases Infeasibility

- *Infeasibility* exists when there is no solution that satisfies all of the problem's constraints.
- In the simplex method, an *infeasible* solution is indicated by looking at the final tableau.
- All  $C_j - Z_j$  row entries will be of the proper sign to imply optimality, but an artificial variable will still be in the solution mix.
- A situation with no feasible solution may exist if the problem was formulated improperly.
  - \* Possibly conflicting constraints.

Indication of *infeasibility*:

- even though all  $C_j - Z_j$ 
  - \* are positive or 0 (the criterion for an optimal solution in a *minimization* case) an artificial variable remains in the solution mix, or
  - \* are negative or 0 (the criterion for an optimal solution in a *maximization* case) all non-slack variables are already in the basis.

## Special Cases

### Unbounded Solutions

- *Unboundedness* describes linear programs that do not have finite solutions.
- It occurs in maximization problems, for example, when a solution variable can be made infinitely large without violating a constraint.
- In the simplex method, the condition of *unboundedness* will be discovered prior to reaching the final tableau.
- The problem will be manifested when trying to decide which variable to remove from the solution mix.
- As seen earlier, the procedure is to divide each quantity column number by the corresponding pivot column number.
  - \* The row with the smallest positive ratio is replaced.
  - \* But if all the ratios turn out to be negative or undefined, it indicates that the problem is unbounded.

## Special Cases Degeneracy

- *Degeneracy* is another situation that can occur when solving an LP problem using the simplex method.
- It develops when three constraints pass through a single point.
- **For example,**
  - o suppose a problem has only these three constraints:  
 $X_1 \leq 10$ ,  $X_2 \leq 10$ , and  $X_1 + X_2 \leq 20$
  - o All three constraint lines will pass through the point (10, 10).
- *Degeneracy* is first recognized when the ratio calculations are made.
- If there is a *tie* for the smallest ratio, this is a signal that *degeneracy* exists.
- As a result of this, when the next tableau is developed, one of the variables in the solution mix will have a value of zero.

## Special Cases Degeneracy

- Degeneracy could lead to a situation known as *cycling*, in which the simplex algorithm alternates back and forth between the same nonoptimal solutions;
  - \* that is, it puts a new variable in, then takes it out in the next tableau, puts it back in, and so on.
- One simple way of dealing with the issue is to select either row in question arbitrarily.
- If unlucky and cycling does occur, simply go back and select a competing row.

## Special Cases Multiple Optimal Solutions

- Multiple, or alternate, optimal solutions can be spotted when the simplex method is being used by looking at the final tableau.
- If the  $C_j - Z_j$  value is equal to 0 for a variable that is not in the solution mix, more than one optimal solution exists.

## Sensitivity Analysis

- *Sensitivity Analysis* shows how the optimal solution and the value of its objective function change, given changes in various inputs to the problem.
- Computer programs handling LP problems of all sizes provide *sensitivity analysis* as an important output feature.
- Those programs use the information provided in the final simplex tableau to compute ranges for the objective function coefficients and ranges for the RHS values.
- They also provide “*shadow prices*,” a concept that will be discussed.

## Sensitivity Analysis High Note Sound Company

*Example from Chapter 7*

maximize profit =  $\$50X_1 + \$120X_2$

subject to:

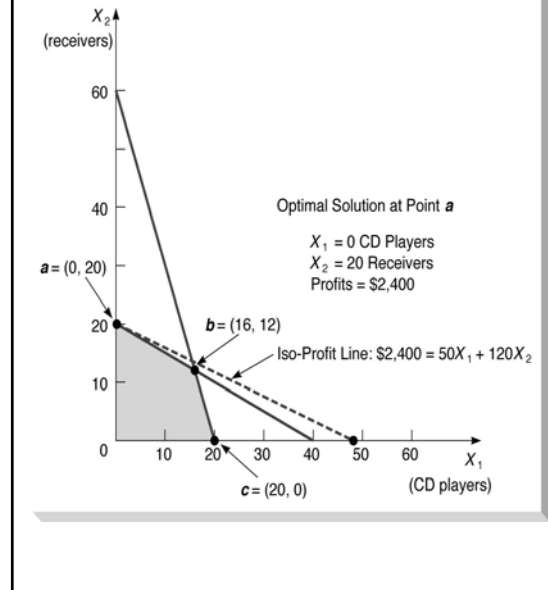
$2X_1 + 4X_2 \leq 80$  (hours of electrician time)

$3X_1 + 1X_2 \leq 60$  (hours of technician time)

Optimal solution:

$X_2 = 20$ receivers	} <i>Basic</i> <i>variables</i>
$S_2 = 40$ hours slack in technician time	
$X_1 = 0$ CD players	} <i>Nonbasic</i> <i>variables</i>
$S_1 = 0$ hours slack in electrician time	

## Sensitivity Analysis High Note Sound Company



## Simplex Solution High Note Sound Company

$C_j$	→	50	120	0	0	
↓	Solution Mix	$X_1$	$X_2$	$S_1$	$S_2$	Quantity
120	$X_2$	1/2	1	1/4	0	20
0	$S_2$	5/2	0	-1/4	1	40
	$Z_j$	60	120	30	0	2400
	$C_j - Z_j$	-10	0	-30	0	

The solution is optimal as long as all  $C_j - Z_j \leq 0$



## Nonbasic Objective Function Coefficients

**Goal:** How much would the Objective function coefficients have to change before one of the nonbasic variables would enter the solution mix and replace one of the basic variables?

$C_j$		50	120	0	0	
	Solution Mix	$X_1$	$X_2$	$S_1$	$S_2$	Quantity
120	$X_2$	1/2	1	1/4	0	20
0	$S_2$	5/2	0	-1/4	1	40
	$Z_j$	60	120	30	0	2400
	$C_j - Z_j$	-10	0	-30	0	

The answer lies in the  $C_j - Z_j$  row of the final simplex tableau.

## Nonbasic Objective Function Coefficients

- In a maximization problem, the basis will not change unless the  $C_j - Z_j$  value of one of the nonbasic variables becomes positive.
  - o The current solution will be optimal as long as all numbers in the bottom row are 0.
  - o The solution is NOT optimal if  $X_1$ 's  $C_j - Z_j$  value is  $> 0$ , or if  $S_1$ 's  $C_j - Z_j$  value is  $< 0$ .
- The values of  $C_j$  for  $X_1$  that do not bring about any change in the optimal solution are given by  $C_j - Z_j \leq 0$ , or  $C_j \leq Z_j$ .

## Nonbasic Objective Function Coefficients

- In this example:
  - For  $X_1$ ,  $C_1 = \$50$  and  $Z_1 = \$60$  meaning current solution is optimal given the profit per CD increases less than \$10 and is  $< \$60$
- In both cases, when maximizing an objective, the value of  $C_j$  may be increased up to the value of  $Z_j$ .
- The value of  $C_j$  for nonbasic variables can be decreased to negative infinity without affecting the solution.
  - The range over which  $C_j$  rates for nonbasic variables can vary without causing a change in the optimal solution mix is called the range of insignificance:
    - ✓  $-\infty \leq C_j \text{ (for } X_1) \leq \$60$

## Simplex Solution High Note Sound Company

$C_j$	→	50	120	0	0	
↓	Sol Mix	$X_1$	$X_2$	$S_1$	$S_2$	Qty
120	$X_1$	$\frac{1}{2}$	1	$\frac{1}{4}$	0	20
0	$S_2$	$\frac{5}{2}$	0	$-\frac{1}{4}$	1	40
	$Z_j$	60	120	30	0	40
	$C_j - Z_j$	0	0	-30	0	2400

Solution mix changes if  $X_1$  profit is greater than 60.

## Basic Objective Function Coefficients

- A change in the profit or cost of a basic variable can affect the  $C_j - Z_j$  values of *all* nonbasic variables because this  $C_j$  is not only in the  $C_j$  row but also in the  $C_j$  column.
  - o This impacts the  $Z_j$  row.
- Consider changing the profit contribution of stereo receivers in this problem.
- Currently, the objective function coefficient is \$120.
- The change in this value can be denoted by the Greek capital letter delta ( $\Delta$ ).
- The final simplex tableau is reworked on the following slide.

## Basic Objective Function Coefficients

$C_j$	→	50	120	0	0	
↓	Solution Mix	$X_1$	$X_2$	$S_1$	$S_2$	Quantity
<b>120</b> <b>+ <math>\Delta</math></b>	$X_1$	1/2	1	1/4	0	20
0	$S_2$	5/2	0	-1/4	1	40
	$Z_j$	60+ $\Delta/2$	120+ $\Delta$	30+ $\Delta/4$	0	2400+20 $\Delta$
	$C_j - Z_j$	-10- $\Delta/2$	0	-30- $\Delta/4$	0	

- The new  $C_j - Z_j$  values for nonbasic variables  $X_1$  and  $S_1$  were determined in exactly the same way as earlier.
- But wherever the  $C_j$  value for  $X_2$  of \$120 was seen before, a new value of  $120 + \Delta$  is used.

## Changes in Resources or RHS Values

- Making changes in the RHS values of constraints (in this example, the resources of electricians' and audio technicians' time) result in changes in the feasible region and often the optimal solution.
- The *shadow price* is the value of one additional unit of a scarce resource.
- *Shadow pricing* provides an important piece of economic information.

## Changes in Resources or RHS Values


- The worth of additional resources is valuable information to management.
- For example, questions as the following can be useful to management:
  - o Exactly how much should a firm be willing to pay to make additional resources available?
  - o Is one more hour of machine time worth \$1 or \$5 or \$20?
  - o Is it worthwhile to pay workers an overtime rate to stay one extra hour each night to increase production output?

## Changes in Resources or RHS Values

- This information is available by looking at the final simplex tableau of an LP problem.
- An important property of the  $C_j - Z_j$  row is that the negatives of the numbers in its slack variable ( $S_i$ ) columns provide us with what we call shadow prices.
- A *shadow price* is the change in value of the objective function from an increase of one unit of a scarce resource.
  - (e.g., by making one more hour of machine time or labor time or other resource available).

## Simplex Solution High Note Sound Company

$C_j$	→	50	120	0	0	
↓	Sol Mix	$X_1$	$X_2$	$S_1$	$S_2$	Qty
120	$X_1$	$\frac{1}{2}$	1	$\frac{1}{4}$	0	20
0	$S_2$	$\frac{5}{2}$	0	$-\frac{1}{4}$	1	40
	$Z_j$	60	120	30	0	40
	$C_j - Z_j$	0	0	-30	0	2400


 Objective increases by 30 if 1 additional hour of electrician time is available.

## Changes in Resources or RHS Values

### *Interpreting the Results*

- The final tableau indicates that the optimal solution is  $X_1 = 0$ ,  $X_2 = 20$ ,  $S_1 = 0$ , and  $S_2 = 40$  and that profit = \$2,400.
  - $S_1$  represents slack availability of the electricians' resource and  $S_2$  the unused time in the audio technicians' department.
- The firm is considering hiring an extra electrician on a part-time basis.
  - Assume it costs \$22 per hour in wages and benefits to bring the part-timer on board.

### **Should the firm do this?**

- The answer is *yes*;
  - the shadow price of the electrician time resource is \$30.
- The firm will *net* \$8 (= \$30 - \$22) for every hour the new worker helps in the production process.

## Changes in Resources or RHS Values

### *Interpreting the Results*

### **Should High Note also hire a part-time audio technician at a rate of \$14 per hour?**

- The answer is *no*;
  - the shadow price is \$0, implying no increase in the objective function by making more of this second resource available.
- Why?
- Because not all of the resource is currently being used—40 hours are still available.
- It would hardly pay to buy more of the resource.

## Right-Hand-Side Ranging

- The range over which shadow prices remain valid is called *right-hand-side ranging*.
- *Ranging* employs the simplex process to find the minimum ratio for a new variable.
- Looking at  $S_1$  for ranging, ratios are calculated from the  $S_1$  column and quantity:
  - Smallest positive ratio =  $20/(1/4) = 80$
  - Smallest negative ratio =  $40/(-1/4) = -160$

$C_j$	→	50	120	0	0		
	↓	Solution Mix	$X_1$	$X_2$	$S_1$	$S_2$	Quantity
120		$X_2$	1/2	1	1/4	0	20
0		$S_2$	5/2	0	-1/4	1	40
		$Z_j$	60	120	30	0	2400
		$C_j - Z_j$	-10	0	-30	0	

## Right-Hand-Side Ranging

- The *smallest positive ratio* (80 in this example) tells by how many hours the electricians' time resource can be *reduced* without altering the current solution mix.
- The RHS resource can be reduced by as much as 80 hours,
  - ✓ basically from the current 80 hours all the way down to 0 hours,
 without causing a basic variable to be pivoted out of the solution.
- The *smallest (in absolute value) negative ratio* (160) tells us the number of hours that can be added to the resource before the solution mix changes.
- Here, electricians' time can be increased by 160 hours, up to 240 (= 80 currently + 160 may be added) hours.
- The range of electricians' time over which the shadow price of \$30 is proved valid.
  - ✓ That range is from 0 to 240 hours.

## Right-Hand-Side Ranging

- The audio technician resource is slightly different in that all 60 hours of time originally available have not been used.
  - $S_2 = 40$  hours in final tableau.
- Applying the ratio test, the number of audio technicians' hours can be reduced by only 40 (the smallest positive ratio =  $40/1$ ) before a shortage occurs.
- Since not all the hours currently available are being used, they can be increased indefinitely without altering the problem's solution.
  - There are no negative substitution rates in the  $S_2$  column, so there are no negative ratios.
- So, the valid range for *this* shadow price would be from 20 (=  $60 - 40$ ) hours to an unbounded upper limit.

## Sensitivity Analysis by Computer

- Solver in Excel has the capability of producing sensitivity analysis that includes the shadow prices of resources.
- The ranging is performed by Excel and highlighted.
- No need for calculating smallest positive or negative ratios.
- Sensitivity is an option once the Solver program is run for a problem.
  - It is a Report option.



## Steps to Form the Dual

- Every LP primal has a dual which provides useful economic information.

To form the dual, first express the primal as a maximization problem with all  $\leq$  constraints.

- o The dual will be a minimization problem.
- o The right-hand-side values of the primal constraints become the objective coefficients of the dual.
- o The primal objective function coefficients become the right-hand-side of the dual constraints.
- o The transpose of the primal constraint coefficients become the dual constraint coefficients.
- o Constraint  $\leq$  signs become  $\geq$  signs.

## Primal and Dual

Primal:

Dual

$$\text{Max } 50X_1 + 120X_2$$

$$\text{Min: } 80U_1 + 60U_2$$

Subject to:

Subject to:

$$2X_1 + 4X_2 \leq 80$$

$$2U_1 + 3U_2 \geq 50$$

$$3X_1 + 1X_2 \leq 60$$

$$4U_1 + 1U_2 \geq 120$$

## Comparison of the Primal and Dual Optimal Tableaus

$C_j$ →		Solution Quantity		\$50	\$120	\$0	\$0
				$X_1$	$X_2$	$S_1$	$S_2$
↓	Mix						
\$7	$X_2$	20	1/2	1	1/4	0	
\$5	$S_2$	40	5/2	0	-1/4	1	
	$Z_j$	\$2,400	60	120	30	0	
	$C_j - Z_j$		-10	0	-30	0	

$C_j$ →		Solution Quantity		\$0	\$0	$M$	$M$	
				$X_1$	$X_2$	$S_1$	$S_2$	$A_1$
↓	Mix							
\$7	$U_1$	30	1	1/4	0	-1/4	0	1/2
\$5	$S_1$	10	0	-5/2	1	-1/2	-1	1/2
	$Z_j$	\$2,400	80	20	0	-20	0	40
	$C_j - Z_j$		\$0	40	0	20	$M$	$M-40$