Basis

A vector $\mathbf{x}$ can be written as a linear combination of its basis elements:

$$\mathbf{x} = \begin{pmatrix} 3 \\ 2 \end{pmatrix} = 3\begin{pmatrix} 1 \\ 0 \end{pmatrix} + 2\begin{pmatrix} 0 \\ 1 \end{pmatrix} = 3\mathbf{e}_1 + 2\mathbf{e}_2$$

Multiplying $\mathbf{x}$ by a matrix $\mathbf{A}$ can be interpreted as a linear combination of $\mathbf{A}\mathbf{e}_1$ and $\mathbf{A}\mathbf{e}_2$.

$$\mathbf{A}\mathbf{x} = \mathbf{A}\begin{pmatrix} 3 \\ 2 \end{pmatrix} = \mathbf{A}\left[3\mathbf{e}_1 + 2\mathbf{e}_2\right] = 3\mathbf{A}\mathbf{e}_1 + 2\mathbf{A}\mathbf{e}_2$$

Imagine that there is a basis $\mathbf{v}_1$ and $\mathbf{v}_2$ such that $\mathbf{A}\mathbf{v}_1 = \lambda_1\mathbf{v}_1$ and $\mathbf{A}\mathbf{v}_2 = \lambda_2\mathbf{v}_2$ (where λ_1 and λ_2 are scalars) and that $\mathbf{x} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2$. Then:

$$\mathbf{A}\mathbf{x} = \mathbf{A}\left[c_1\mathbf{v}_1 + c_2\mathbf{v}_2\right] = c_1\mathbf{A}\mathbf{v}_1 + c_2\mathbf{A}\mathbf{v}_2 = c_1\lambda_1\mathbf{v}_1 + c_2\lambda_2\mathbf{v}_2$$

If $\mathbf{A}\mathbf{v} = \lambda\mathbf{v}$, call λ an **eigenvalue** of $\mathbf{A}$ and $\mathbf{v}$ an **eigenvector** of $\mathbf{A}$.

Power Method

Suppose a matrix $\mathbf{A}$ has two distinct, real eigenvalues λ_1 and λ_2 with eigenvectors $\mathbf{x}_1$ and $\mathbf{x}_2$. If $\mathbf{x}_1$ and $\mathbf{x}_2$ are not parallel, you can write any vector $\mathbf{z}^{(0)}$ as a linear combination of $\mathbf{x}_1$ and $\mathbf{x}_2$:

$$\mathbf{z}^{(0)} = c_1\mathbf{x}_1 + c_2\mathbf{x}_2$$

Consider fixed point iteration $\mathbf{z}^{(k+1)} = \mathbf{A}\mathbf{z}^{(k)}$ so that $\mathbf{z}^{(k)} = \mathbf{A}^k\mathbf{z}^{(0)}$:

$$\begin{aligned}
\mathbf{z}^{(1)} &= \mathbf{A}\mathbf{z}^{(0)} = c_1\mathbf{A}\mathbf{x}_1 + c_2\mathbf{A}\mathbf{x}_2 = c_1\lambda_1\mathbf{x}_1 + c_2\lambda_2\mathbf{x}_2 \\
\mathbf{z}^{(2)} &= \mathbf{A}\mathbf{z}^{(1)} = c_1\lambda_1\mathbf{A}\mathbf{x}_1 + c_2\lambda_2\mathbf{A}\mathbf{x}_2 = c_1\lambda_1^2\mathbf{x}_1 + c_2\lambda_2^2\mathbf{x}_2 \\
\mathbf{z}^{(k)} &= \mathbf{A}\mathbf{z}^{(k-1)} = c_1\lambda_1^{k-1}\mathbf{A}\mathbf{x}_1 + c_2\lambda_2^{k-1}\mathbf{A}\mathbf{x}_2 = c_1\lambda_1^k\mathbf{x}_1 + c_2\lambda_2^k\mathbf{x}_2
\end{aligned}$$

Now, if $|\lambda_1| > |\lambda_2|$, then $\left(\frac{\lambda_2}{\lambda_1}\right)^k \rightarrow 0$ as $k \rightarrow \infty$. Therefore:

$$\mathbf{z}^{(k)} = \lambda_1^k \left(c_1 \mathbf{x}_1 + c_2 \left(\frac{\lambda_2}{\lambda_1}\right)^k \mathbf{x}_2 \right) \approx c_1 \lambda_1^k \mathbf{x}_1 \quad \text{as} \quad k \rightarrow \infty$$

But if $\mathbf{z}^{(k)} \sim \lambda_1^k$, then $\frac{|\mathbf{z}^{(k)}|}{|\mathbf{z}^{(k-1)}|} \sim \lambda_1$. One way to look at the ratio of $\mathbf{z}^{(k)}$ to $\mathbf{z}^{(k-1)}$ is to evaluate the Rayleigh coefficient:

$$r^{(k)} = \frac{\mathbf{z}^{(k-1)} \cdot \mathbf{z}^{(k)}}{\mathbf{z}^{(k-1)} \cdot \mathbf{z}^{(k-1)}} \rightarrow \lambda_1 \quad \text{as} \quad k \rightarrow \infty$$

This method can be extended to look for the other eigenvalues of $\mathbf{A}$ and for complex eigenvalues as well. It works most effectively in finding the extreme eigenvalues — those with the largest and smallest magnitude.

Eigenvalues and Eigenvectors

To find the eigenvalues and eigenvectors of a matrix $\mathbf{A}$, we look for solutions to the equation $\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$. (The matrix $\mathbf{A}$ comes from an equations for the evolution of a mass-spring damper system where K represents the stiffness of the spring.)

$$\mathbf{A}\mathbf{x} = \lambda\mathbf{x} \qquad \longleftrightarrow \qquad \begin{pmatrix} -2 & -K \\ 1 & 0 \end{pmatrix}\mathbf{x} = \lambda\mathbf{x}$$

$$\mathbf{A}\mathbf{x} = \lambda\mathbf{I}\mathbf{x} \qquad \longleftrightarrow \qquad \begin{pmatrix} -2 & -K \\ 1 & 0 \end{pmatrix}\mathbf{x} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\mathbf{x}$$

$$(\mathbf{A} - \lambda\mathbf{I})\,\mathbf{x} = 0 \qquad \longleftrightarrow \qquad \begin{pmatrix} -2-\lambda & -K \\ 1 & -\lambda \end{pmatrix}\mathbf{x} = \mathbf{0}$$

Since the last equation is satisfied for a non-zero $\mathbf{x}$, the matrix $(\mathbf{A} - \lambda\mathbf{I})$ is singular:

$$\det\begin{pmatrix} -2-\lambda & -K \\ 1 & -\lambda \end{pmatrix} = 0 \quad \text{and} \quad (-2-\lambda)(-\lambda) + K = 0$$

This polynomial is called the **characteristic polynomial** of the matrix $\mathbf{A}$ and the roots λ of the characteristic polynomial are the **eigenvalues** of the matrix $\mathbf{A}$. The vector $\mathbf{x}$ is an **eigenvector** of the matrix $\mathbf{A}$.

We have the polynomial $\lambda^2 + 2\lambda + K = 0$. First consider the case where $K = 0$: couple of cases:

$$K = 0 \qquad \Rightarrow \quad \lambda = \frac{-2 \pm \sqrt{4}}{2} = -1 \pm 1 = 0, -2$$

The eigenvalues for $K = 0$ are $\lambda = -2, 0$. Remember that the eigenvalue equation may be written $\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$. Look for the eigenvector $\mathbf{x}$ corresponding to each eigenvalue λ:

$$\mathbf{A}\mathbf{x}_1 = \lambda_1\mathbf{x}_1 \qquad \Rightarrow \qquad \begin{pmatrix} -2 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = -2 \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$-2x_1 = -2x_1 \qquad \text{and} \quad x_1 = -2x_2 \quad \Rightarrow \quad \mathbf{x}_1 = \begin{pmatrix} 1 \\ -0.5 \end{pmatrix}$$

$$\mathbf{A}\mathbf{x}_2 = \lambda_2\mathbf{x}_2 \qquad \Rightarrow \qquad \begin{pmatrix} -2 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0 \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$-2x_1 = 0 \qquad \text{and} \quad x_1 = 0 \quad \Rightarrow \quad \mathbf{x}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Note that the eigenvectors are only determined up to a constant: $\mathbf{A}(3\mathbf{x}) = \lambda(3\mathbf{x})$.

Eigenvalues of real matrices come in two different flavors:

• Purely real: correspond to stretching/contraction or to growth/decay in time-dependent problems.

• Complex conjugate pairs (i.e. $\lambda_r \pm \sqrt{-1}\lambda_i$): correspond to rotation or oscillation in time-dependent problems. The real part can also lead to stretching/contraction and to growth/decay in time-dependent problems.

Look at different values of K in our example:

$$K = 1 \qquad \Rightarrow \quad \lambda = \frac{-2 \pm \sqrt{0}}{2} = -1 \pm 0 = -1, -1$$

$$K = 2 \qquad \Rightarrow \quad \lambda = \frac{-2 \pm \sqrt{-4}}{2} = -1 \pm \sqrt{-1}$$

The real part of the eigenvalues in these case is negative, so that solutions that evolve as $\exp(\lambda t)$ will decay with time.

Examples/Applications

Error analysis of iterative methods: The initial error is multiplied repeatedly by a matrix $\mathbf{T}$:

$$\begin{aligned}
\mathbf{e}^{(1)} &= \mathbf{T}\,\mathbf{e}^{(0)} = \mathbf{T}\,(c_1\mathbf{x}_1 + c_2\mathbf{x}_2) = c_1\mathbf{T}\,\mathbf{x}_1 + c_2\mathbf{T}\,\mathbf{x}_2 = c_1\lambda_1\mathbf{x}_1 + c_2\lambda_2\mathbf{x}_2 \\
\mathbf{e}^{(k)} &= \mathbf{T}^k\,\mathbf{e}^{(0)} = c_1(\lambda_1)^k\mathbf{x}_1 + c_2(\lambda_2)^k\mathbf{x}_2
\end{aligned}$$

Therefore, if $|\lambda_1| < 1$ and $|\lambda_2| < 1$ then the error will go to zero as $k \to \infty$.

Ordinary differential equations: Consider a system of ordinary differential equations:

$$\dot{\mathbf{v}} = \mathbf{A}\,\mathbf{v}$$

If $\mathbf{x_1}$ and $\mathbf{x}_2$ are the eigenvectors of $\mathbf{A}$ (i.e. $\mathbf{A}\mathbf{x}_1 = \lambda_1\mathbf{x}_1$ and $\mathbf{A}\mathbf{x}_2 = \lambda_2\mathbf{x}_2$), then:

$$\begin{aligned}
\mathbf{v}(t) &= c_1(t)\mathbf{x}_1 + c_2(t)\mathbf{x}_2 \\
\text{and} \qquad \dot{\mathbf{v}} &= \dot{c}_1\mathbf{x}_1 + \dot{c}_2\mathbf{x}_2 = \mathbf{A}\,\mathbf{v} = c_1\mathbf{A}\mathbf{x}_1 + c_2\mathbf{A}\mathbf{x}_2 = c_1\lambda_1\mathbf{x}_1 + c_2\lambda_2\mathbf{x}_2
\end{aligned}$$

This implies that:

$$[\dot{c}_1 - \lambda_1 c_1]\,\mathbf{x}_1 + [\dot{c}_2 - \lambda_2 c_2]\,\mathbf{x}_2 = 0$$

For this to be true for all values of c_1 and c_2 (including zero), we need:

$$\begin{array}{lcl} \dot{c}_1 & = & \lambda_1 c_1 \\ \dot{c}_2 & = & \lambda_2 c_2 \end{array} \quad \Rightarrow \quad \begin{array}{lcl} c_1(t) & = & c_1(t=0)\ \exp(\lambda_1 t) \\ c_2(t) & = & c_2(t=0)\ \exp(\lambda_2 t) \end{array}$$

The solution to the (system of) ordinary differential equations is:

$$\mathbf{v}(t) = c_1(t=0)\ \exp(\lambda_1 t)\,\mathbf{x}_1 + c_2(t=0)\ \exp(\lambda_2 t)\,\mathbf{x}_2$$

- If the real part of λ_1 and λ_2 is negative, the solutions will decay with time.
- If the real part of λ_1 and λ_2 is positive, the solutions will grow with time.
- If the eigenvalues are complex ($\lambda = \lambda_r + \sqrt{-1}\lambda_i$), the solution will evolve like:

$$\exp(\lambda t) = \exp(\lambda_r t)\left[cos(\lambda_i t) + \sqrt{-1} sin(\lambda_i t)\right]$$