Numerical Solutions to Partial Differential Equations

Introduction

General partial differential equations (PDE) is hard to solve! We shall only treat some special types of PDE's that are useful and easier to be solved.

Classification of 2nd order quasi-linear PDE's

General form

$$a(x,y)\frac{\partial^2 u}{\partial x^2} + 2b(x,y)\frac{\partial^2 u}{\partial x \partial y} + c(x,y)\frac{\partial^2 u}{\partial y^2} = F\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right)$$

quasi-linear — linear in highest order derivatives

$u = u(x,y)$ — unknown functions to be solved.

x, y — independent variable x and y.

Some standard notations

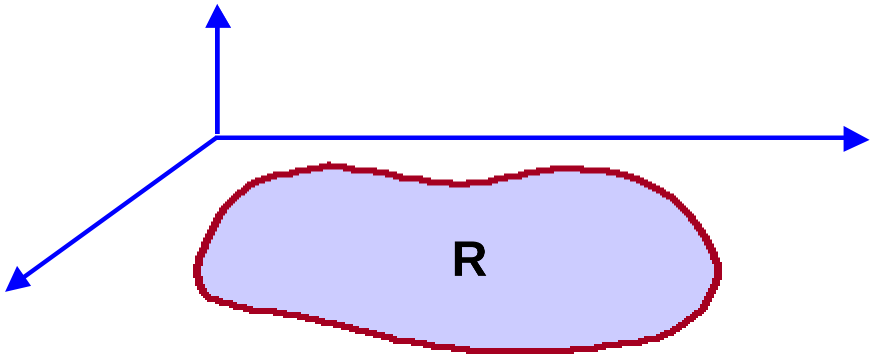
$$u_x := \frac{\partial u}{\partial x}, \quad u_y = \frac{\partial u}{\partial y}, \quad u_{xx} := \frac{\partial^2 u}{\partial x^2}, \quad u_{yy} := \frac{\partial^2 u}{\partial y^2}, \quad u_{xy} := \frac{\partial^2 u}{\partial x \partial y}$$

Types of equations

Type	**Condition**	**Example**	
elliptic	$b^2 - ac < 0$	Laplace equation:	$u_{xx} + u_{yy} = 0$ $\{a = 1, b = 0, c = 1\}$
parabolic	$b^2 - ac = 0$	Heat equation:	$k^2 u_{xx} = u_t$ $\{a = k^2, b = 0, c = 0\}$
hyperbolic	$b^2 - ac > 0$	Wave equation:	$A^2 u_{xx} = u_{tt}$ $\{a = A^2, b = 0, c = -1\}$

Methods of solutions depended on the type of equations.

Geometrically



Type may not be constant over **R** because a,b,c can vary over **R**, e.g, elliptic in one part of **R** and parabolic in the other part of **R**.

Example:

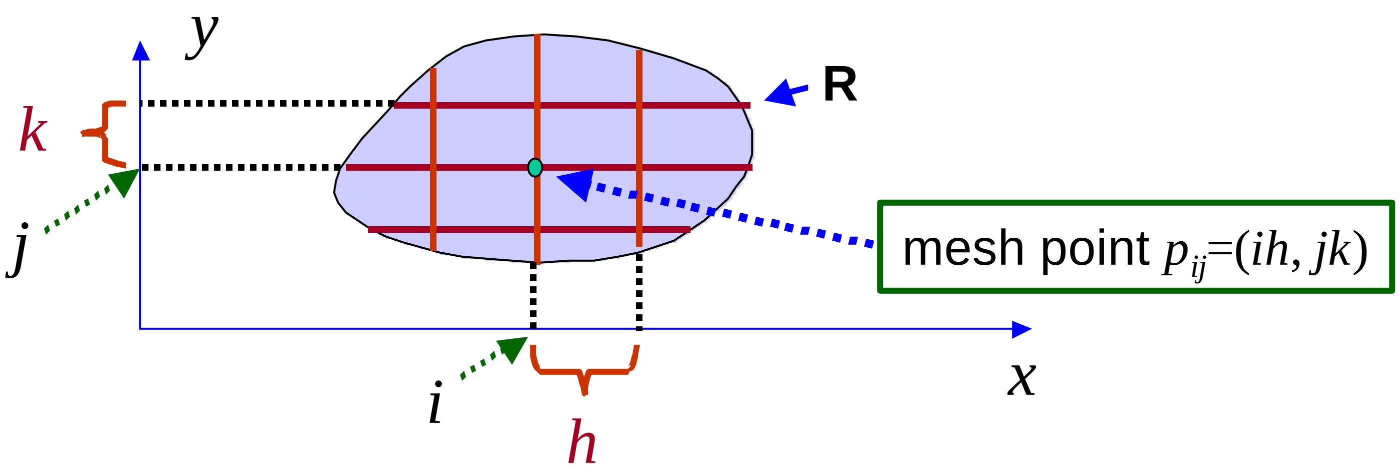
$$\underbrace{(\sin^2 y)}_{a} u_{xx} + u_{yy} = -[u + (\sin y) u_x]$$

$$\mathbf{R}: -3 \le x \le 3, \quad -3 \le y \le 3$$

$$\Rightarrow a = \sin^2 y,\ b = 0,\ c = 1 \Rightarrow b^2 - ac = -\sin^2 y \le 0 \begin{cases} = 0 & \textbf{parabolic} \\ < 0 & \textbf{elliptic} \end{cases}$$

General Approach to the Solutions of PDEs

Step 1: Define a grid on **R** with "mesh points"



Step 2: Approximate derivatives at mesh points by central difference quotients

$$u_x(ih, jk) = \frac{u_{i+1,j} - u_{i-1,j}}{2h}, \quad u_y(ih, jk) = \frac{u_{i,j+1} - u_{i,j-1}}{2k},$$

$$u_{xx}(ih, jk) = \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2}, \quad u_{yy}(ih, jk) = \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{k^2}$$

These will bring a PDE to a difference equation relating u_{ij} to its neighbouring points in the grid.

For example,

$$u_{xx} + u_{yy} = 0 \quad \Rightarrow \quad \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2} + \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{k^2} = 0$$

$$\boxed{k^2 u_{i+1,j} + k^2 u_{i-1,j} + h^2 u_{i,j+1} + h^2 u_{i,j-1} - 2(k^2 + h^2) u_{i,j} = 0}$$

Step 3: Arrange the resulting difference equation into a system of linear equations

$$\begin{bmatrix} * & * & \cdots & * \\ * & * & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \cdots & * \end{bmatrix} \begin{bmatrix} u_{11} \\ u_{12} \\ \vdots \\ * \end{bmatrix} = \begin{bmatrix} * \\ * \\ \vdots \\ * \end{bmatrix}$$

Taking into consideration of boundary conditions and solve it for $u_{11}, u_{12}, \ldots$

Step 4: change grid size for a more accurate approximation.

$$h \to \frac{h}{2} \to \frac{h}{4} \cdots\cdots \qquad k \to \frac{k}{2} \to \frac{k}{4} \cdots\cdots$$

Solution to Elliptic Type's PDE

The general approach will be followed to solve these types of problems by taking into account various kinds of boundary conditions in form of the system of linear equations. We will illustrate this using the following PDE:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = u_{xx} + u_{yy} = f(x, y) \equiv 0$$

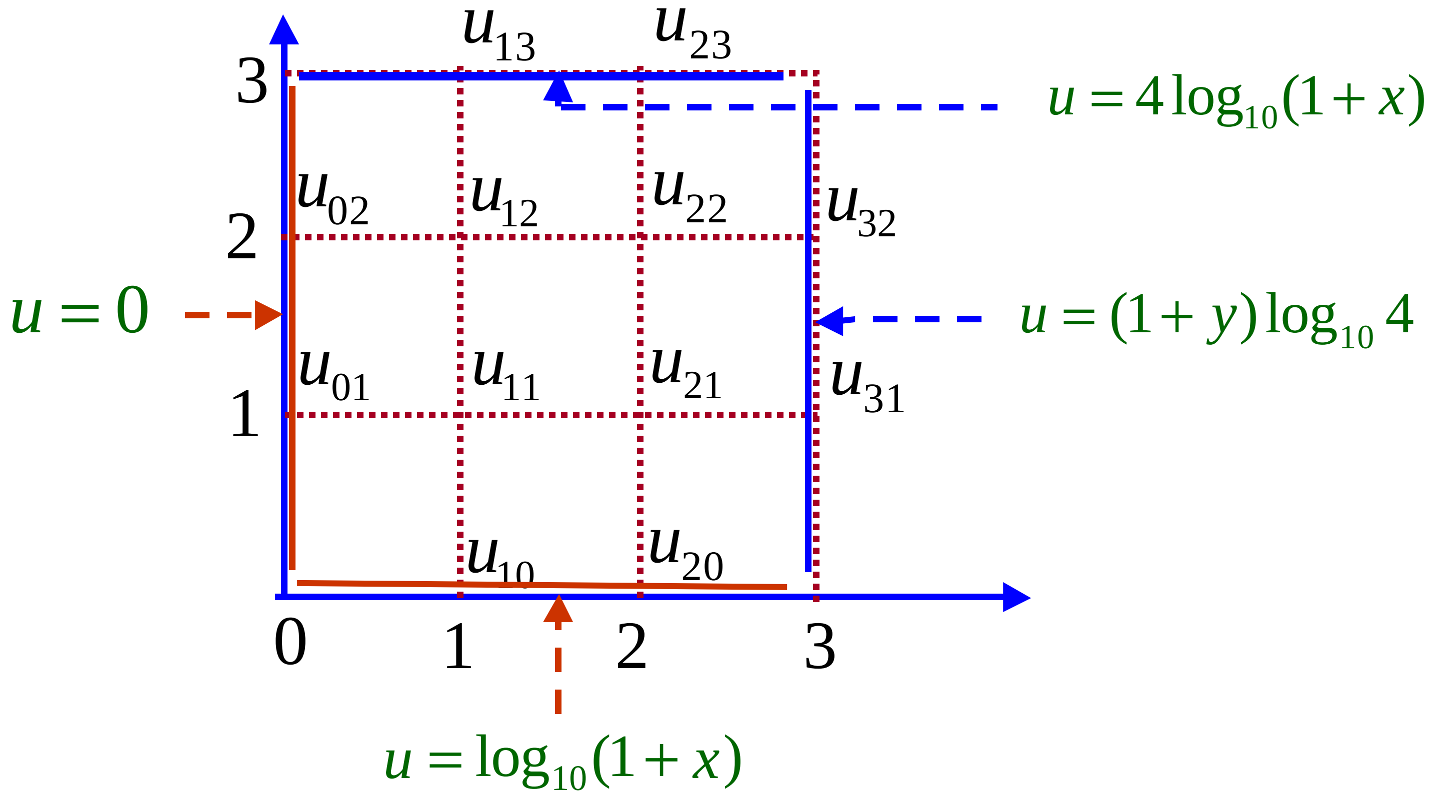
$$\mathbf{R} = \left\{ (x, y),\ 0 \le x \le 3,\ 0 \le y \le 3 \right\}$$

Boundary condition $u(x, y) = (1 + y)\log_{10}(1 + x)$

We follow the step-by-step procedure given in the previous section.

Step 1: Define a grid along with an order of mesh-points inside **R**. (We have to be clear about **R** and h, k)

First, let us start with a crude grid $h = k = 3/N$, for N=3 —> $h = k = 1$



knowns: $u_{13}, u_{23}, u_{32}, u_{31}, u_{02}, u_{01}, u_{10}, u_{20}$ unknowns: $u_{11}, u_{12}, u_{21}, u_{22}$

Step 2: Approximate derivatives at mesh-points

$$u_{xx} + u_{yy} = f(x, y) = 0 \quad \Rightarrow$$

$$\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2} + \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{k^2} = f_{ij} \equiv 0 \quad i = 1,2,3 \quad j = 1,2,3$$

$$\Rightarrow \quad u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j} = 0$$

At mesh-point (i, j) where $u_{i,j}$ is unknown:

@ (1,1): $0 = u_{01} - 2u_{11} + u_{21} + u_{10} - 2u_{11} + u_{12} = -4u_{11} + \underbrace{u_{01}}_{0} + u_{21} + \underbrace{u_{10}}_{0.301} + u_{12}$

@ (2,1): $0 = u_{11} + \underbrace{u_{31}}_{1.204} + \underbrace{u_{20}}_{0.477} + u_{22} - 4u_{21}$

@ (1,2): $0 = u_{11} + \underbrace{u_{13}}_{1.204} + \underbrace{u_{02}}_{0} + u_{22} - 4u_{12}$

@ (2,2): $0 = u_{21} + \underbrace{u_{23}}_{1.908} + u_{12} + \underbrace{u_{32}}_{1.806} - 4u_{22}$

Boundary values are known

Step 3: Arrange the equation into matrix form

$$\begin{pmatrix} -4 & 1 & 1 & 0 \\ 1 & -4 & 0 & 1 \\ 1 & 0 & -4 & 1 \\ 0 & 1 & 1 & -4 \end{pmatrix} \begin{pmatrix} u_{12} \\ u_{22} \\ u_{11} \\ u_{21} \end{pmatrix} = \begin{pmatrix} -1.204 \\ -3.714 \\ -0.301 \\ -1.681 \end{pmatrix}$$

Solve the equations for

$$\begin{pmatrix} u_{12} \\ u_{22} \\ u_{11} \\ u_{21} \end{pmatrix} = \begin{pmatrix} 0.756 \\ 1.336 \\ 0.483 \\ 0.875 \end{pmatrix}$$

Step 4: Refine the step-size by choosing smaller h, k.

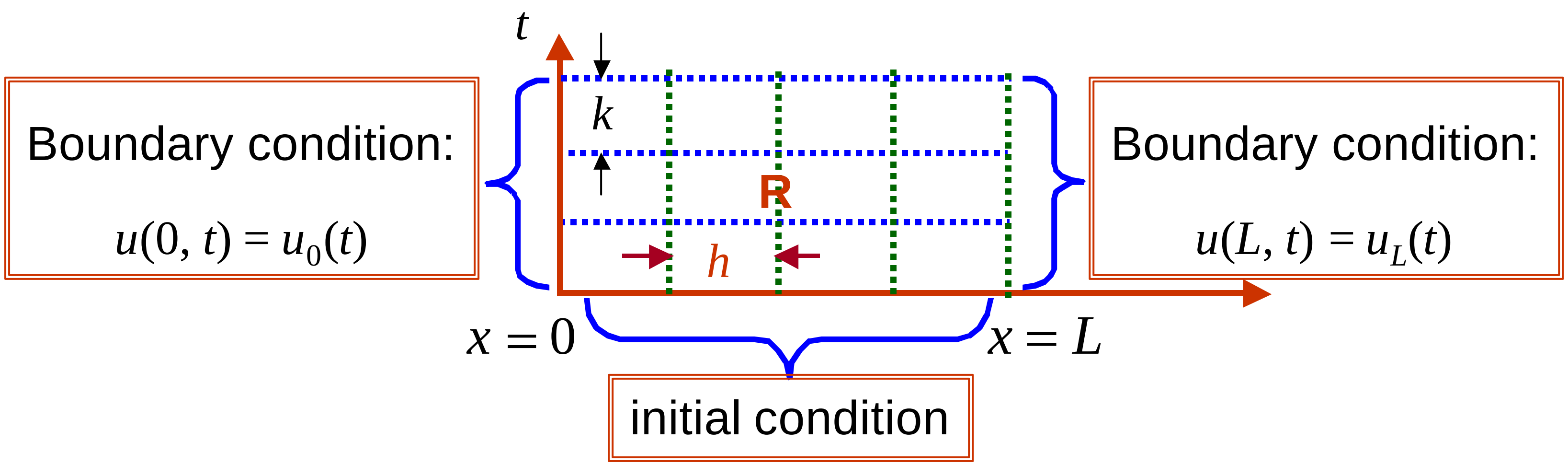
Parabolic and Hyperbolic Types

Parabolic: Example — heat equation

$$Du_{xx} = u_t, \qquad \textbf{where } D \textbf{ is heat diffusion coefficient}$$

Hyperbolic: Example — wave equation

$$C^2 u_{xx} = u_{tt} \qquad \text{where } C^2 \text{ is wave propagation velocity}$$



We will use parabolic type $u_{xx} = u_t$ to illustrate the solution method,

which carries over to the hyperbolic type as well!

Notations:

$$x_i = i \cdot h,\ \ i = 0, 1, \cdots, N, \quad L = Nh \ \Rightarrow \ h = \frac{L}{N}$$

$$t_j = j \cdot k,\ \ j = 0, 1, \cdots$$

$$u_{i,j} = u(x_i, t_j)$$

$$u_t(x_i, t_j) = \frac{u(x_i, t_{j+1}) - u(x_i, t_j)}{k} = \frac{u_{i,j+1} - u_{i,j}}{k}$$

$$u_{xx}(x_i, t_j) = \frac{u(x_{i+1}, t_j) - 2u(x_i, t_j) + u(x_{i-1}, t_j)}{h^2} = \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2}$$

Then

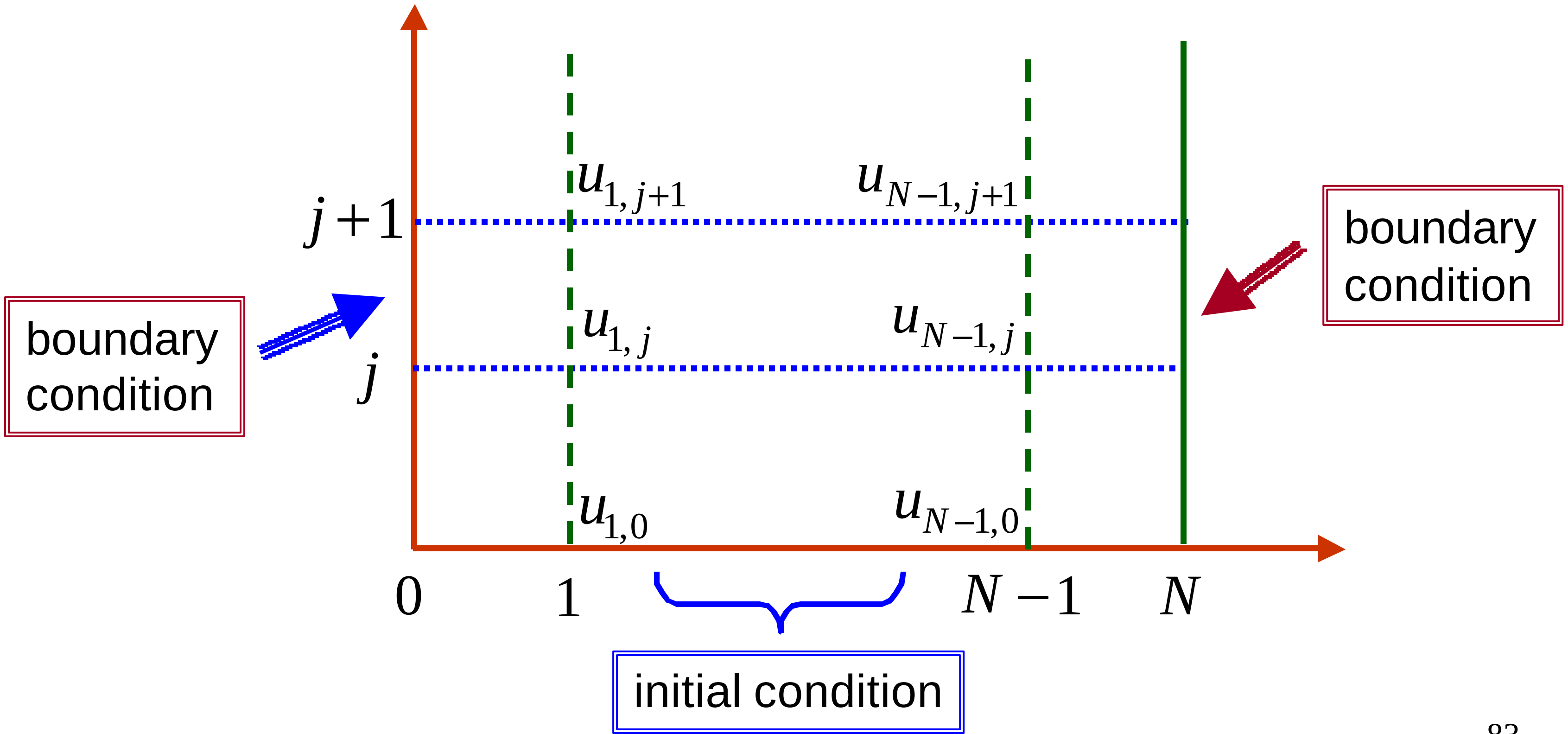
$$u_t = D\ u_{xx} \quad \Rightarrow \quad u_{i,j+1} - u_{i,j} = \frac{kD}{h^2}\left(u_{i+1,j} - 2u_{i,j} + u_{i-1,j}\right) \qquad (\clubsuit)$$

To solve the equation, we start with $j = 0$, then $u_{i,0}$'s are given as initial conditions and can be used to solve for $u_{i,1}, \ \ i = 1, \cdots, N-1$

Rewrite equation (♣) as

$$u_{i,j+1} = g \cdot u_{i+1,j} + (1-2g)u_{i,j} + g \cdot u_{i-1,j} \qquad \text{with } g = \frac{kD}{h^2}$$

In general, we can solve for $u_{i,j+1}$, $i = 1, \ldots, N$, if we know the j-th row.



Example: Solve the following boundary value problem,

$$u_t = u_{xx}, \quad 0 \le x \le 1, \quad D = 1, \quad L = 1$$

initial condition: $u(x,0) = \sin \frac{px}{2}$ boundary condition: $u(0,t) = 0, \; u(1,t) = 1$

We choose $N = 3$ and hence $h = 1/3$ and choose two different k:

$k = 0.05 \mapsto g = 0.45$

$jk \setminus i$	$u_{0,j}$	$u_{1,j}$	$u_{2,j}$	$u_{3,j}$
0.00	0	0.500	0.866	1
0.05	0	0.434	0.762	1
0.10	0	0.387	0.724	1
0.15	0	0.364	0.696	1
0.20	0	0.350	0.684	1
0.25	0	0.343	0.676	1
0.30	0	0.338	0.672	1
0.35	0	0.336	0.667	1
0.40	0	0.335	0.668	1
0.45	0	0.334	0.668	1
0.50	0	0.333	0.667	1

$k = 0.1 \mapsto g = 0.9$

$jk \setminus i$	$u_{1,j}$	$u_{2,j}$
0.0	0.500	0.866
0.1	0.379	0.657
0.2	0.288	0.716
0.3	0.414	0.587
0.4	0.197	0.803
0.5	0.505	0.435
0.6	−0.061	1.061
0.7	1.003	−0.003
0.8	−0.804	1.805
0.9	2.269	−1.269
1.0	−2.957	3.457

unstable case (rows 0.1–0.9)

A short discussion about hyperbolic type PDE:

PDE: $u_{tt} = C^2(x,t)u_{xx}, \quad 0 \le x \le 1, \quad t \ge 0$

Initial conditions: $u(x,0) = f_1(x), \quad u_t(x,0) = f_2(x)$

Boundary conditions: $u(0,t) = g_0(t), \quad u(1,t) = g_1(t)$

Following the usual procedure, we obtain an approximation:

$$u_{i,j+1} = \left(2 - 2g^*\right)u_{i,j} + g^* \cdot u_{i-1,j} + g^* \cdot u_{i+1,j} - u_{i,j-1} \qquad \text{with } g^* = \frac{C^2 k^2}{h^2}$$

Note that at $j = 0$, we have to deal with $u_{i,-1}$, which are not readily available.

Thus, we will have to compute these terms first.

$$u_t(x,0) = f_2(x) \quad \Rightarrow \quad u_{i,1} - u_{i,-1} = 2kf_2(x_i) \quad \Rightarrow \quad \boxed{u_{i,-1} = u_{i,1} - 2kf_2(x_i)}$$

The difference equation can then be solved by using the direct method, e.g,

$$u_{i,1} = (2-2g^*)u_{i,0} + g^* u_{i-1,0} + g^* u_{i+1,0} - u_{i,-1}$$
$$= (2-2g^*)u_{i,0} + g^* u_{i-1,0} + g^* u_{i+1,0} - u_{i,1} + 2kf_2(x_i)$$

⇓

$$u_{i,1} = (1-g^*) f_1(x_i) + \frac{1}{2}g^* f_1(x_{i-1}) + \frac{1}{2}g^* f_1(x_{i+1}) + kf_2(x_i), \quad i = 1, 2, \cdots, N-1$$

For $j > 1$, we still use

$$u_{i,j+1} = \left(2-2g^*\right)u_{i,j} + g^* \cdot u_{i-1,j} + g^* \cdot u_{i+1,j} - u_{i,j-1}$$

The rest of computational procedure is exactly the same as that in the parabolic case.

Example: Solve

$$
\begin{aligned}
&PDE: && u_{tt} = u_{xx} \quad 0 \le x \le 1, \quad t \ge 0 \\
&Initial\ conditions: && u(x,0) = f_1(x) = 0 \\
& && u_t(x,0) = f_2(x) = x + \sin(px) \\
&Boundary\ conditions: && u(0,t) = g_0(t) = 0 \\
& && u(1,t) = g_1(t) = \frac{1}{p}\sin(pt)
\end{aligned}
$$

Let us choose $h = k = 0.25$ so that $\gamma^* = 1$

Determine $u_{i,-1}$ to start the solution or use formula on the previous page

to compute $u_{i,1}$, $i = 1, 2, 3$, first, i.e.,

$$
u_{i,1} = 0 + k \cdot f_2(x_i) = 0.25\left[x_i + \sin(\eth x_i)\right] \Rightarrow \begin{cases} u_{1,1} = 0.239 \\ u_{2,1} = 0.375 \\ u_{3,1} = 0.364 \end{cases}
$$

D.I.Y. to complete the solutions up to $t = 2$.