Numerical Integration

Given a function $f(x)$ and an interval, say $[a, b]$, we want to find an algorithm to approximate

$$\int_a^b f(x)dx \cong ?$$

Newton-Cotes Integration Method

Using Newton-Gregory Forward Polynomial

$$P_n(x) = f_0 + s\Delta f_0 + \frac{s(s-1)}{2!}\Delta^2 f_0 + \cdots$$

to interpolate (approximate) $f(x)$ in $[a, b]$, i.e.,

$$\int_a^b f(x)dx \cong \int_a^b P_n(x)dx$$

NEWTON-COTES INTEGRATION

Although the analytical procedure could be used to find out the expression of integrals, a large number of integrals do not have solutions in closed form. Numerical integration applies regardless of the complexity of the integrand or the existence of a closed form for the integral.

General Consideration:

A very simple method used in the numerical integration is the Newton-Cotes forward polynomial, particularly the polynomial of degrees 1, 2 and 3; i.e;

$$\int_a^b f(x)dx \cong \int_a^b P_n(x_s)dx$$

- Let us now develop our three important Newton-Cotes formulas. During the integration, we will need to change the variable of integration from x to s, since our polynomials are expressed in terms of s. Observe that

$$s = \frac{x - x_0}{h} \iff dx = h \cdot ds \implies \left\{ \begin{array}{l} x = x_0 \Updownarrow s = \frac{x - x_0}{h} = 0 \\ x = x_1 = x_0 + h \Updownarrow s = \frac{x_0 + h - x_0}{h} = 1 \end{array} \right.$$

For $n = 1$, we have

$$\int_{x_0}^{x_1} f(x)dx \cong \int_{x_0}^{x_1} (f_0 + s\Delta f_0)dx = h\int_0^1 (f_0 + s\Delta f_0)ds$$

$$= \left[hf_0 s + h\Delta f_0 \frac{s^2}{2} \right]_0^1 = h\left(f_0 + \frac{1}{2}\Delta f_0 \right)$$

$$= \frac{h}{2}[2f_0 + (f_1 - f_0)] = \frac{h}{2}(f_0 + f_1)$$

For $n = 2$, we have

$$\int_{x_0}^{x_2} f(x)dx \cong \int_{x_0}^{x_2}\left(f_0 + s\Delta f_0 + \frac{s(s-1)}{2}\Delta^2 f_0 \right)dx$$

$$= h\int_0^2\left(f_0 + s\Delta f_0 + \frac{s(s-1)}{2}\Delta^2 f_0 \right)ds$$

$$= \left[hf_0 s + h\Delta f_0 \frac{s^2}{2} + h\Delta^2 f_0\left(\frac{s^3}{6} - \frac{s^2}{4} \right) \right]_0^2$$

$$= h\left(2f_0 + 2\Delta f_0 + \frac{1}{3}\Delta^2 f_0 \right)$$

$$= \frac{h}{3}\left(f_0 + 4f_1 + f_2 \right)$$

$$x = x_2 = x_0 + 2h$$

$$\Updownarrow$$

$$s = \frac{x_0 + 2h - x_0}{h} = 2$$

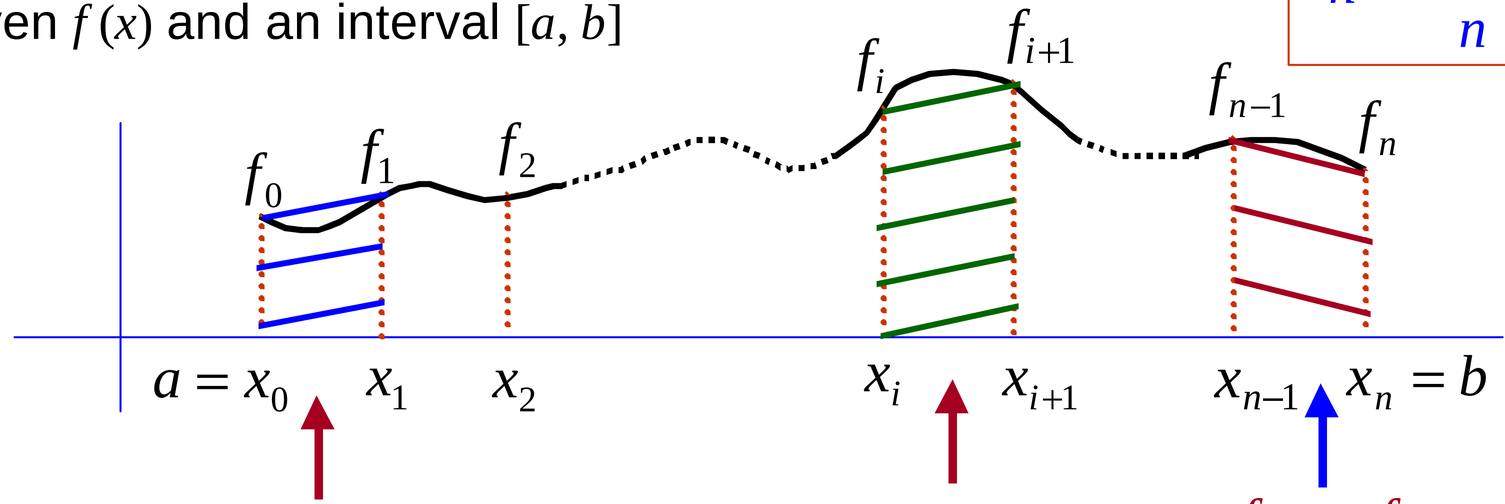
For $n = 3$, we have

$$\int_{x_0}^{x_3} f(x)dx \cong \frac{3h}{8}\left(f_0 + 3f_1 + 3f_2 + f_3 \right)$$

The Trapezoidal Rule for Integration

Given $f(x)$ and an interval $[a, b]$

$$h = \frac{b-a}{n}$$



$$h\frac{f_0 + f_1}{2} \qquad h\frac{f_i + f_{i+1}}{2} \qquad h\frac{f_{n-1} + f_n}{2}$$

Thus,

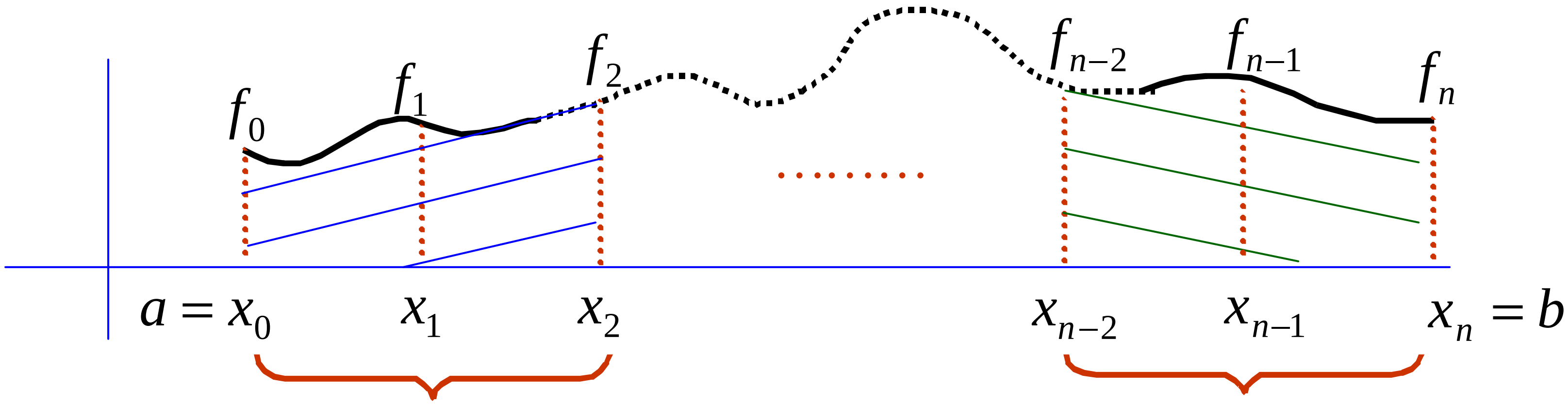
$$\int_a^b f(x)dx \cong h\frac{f_0 + f_1}{2} + h\frac{f_1 + f_2}{2} + \cdots + h\frac{f_i + f_{i+1}}{2} + \cdots + h\frac{f_{n-1} + f_n}{2}$$

$$= \frac{h}{2}\left(f_0 + 2f_1 + 2f_2 + \cdots + 2f_{n-1} + f_n\right)$$

The Simpson's $\frac{1}{3}$ Rule for Integration

$$h = \frac{b-a}{n}, \quad n \text{ is even}$$

Given $f(x)$ and an interval $[a, b]$



Newton-Cotes: $\frac{h}{3}(f_0 + 4f_1 + f_2)$ .... $\frac{h}{3}(f_{n-2} + 4f_{n-1} + f_n)$

Thus,

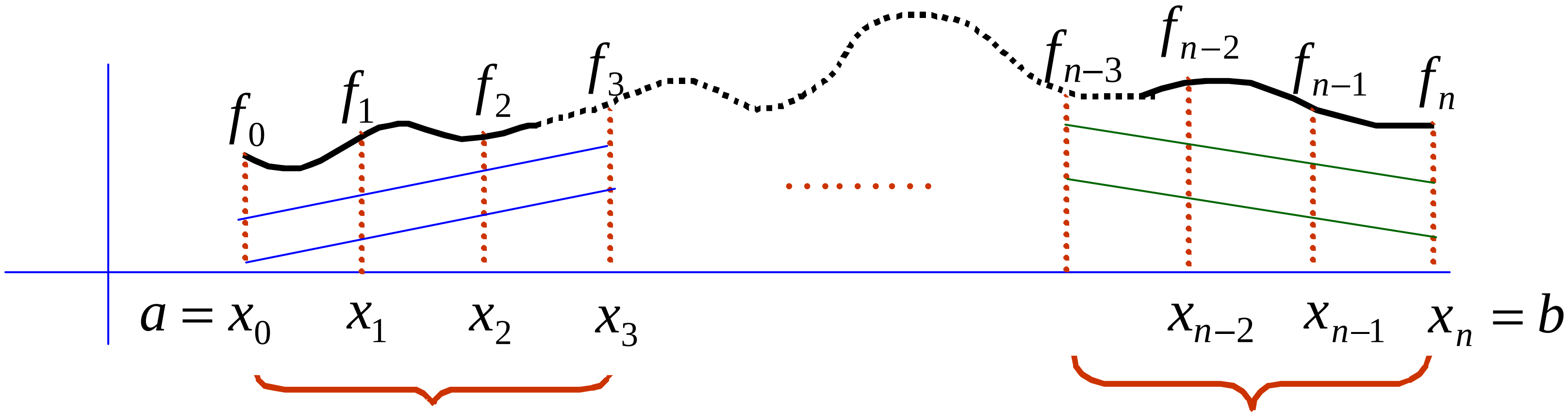
$$\int_a^b f(x)dx \cong \frac{h}{3}(f_0 + 4f_1 + f_2) + \cdots + \frac{h}{3}(f_{n-2} + 4f_{n-1} + f_n)$$

$$= \frac{h}{3}(f_0 + 4f_1 + 2f_2 + 4f_3 + 2f_4 + \cdots + 2f_{n-2} + 4f_{n-1} + f_n)$$

The Simpson's $\frac{3}{8}$ Rule for Integration

$$h = \frac{b-a}{n}, \quad n = 3m$$

Given $f(x)$ and an interval $[a, b]$



Newton-Cotes: $\frac{3h}{8}\left(f_0 + 3f_1 + 3f_2 + f_3\right)$ ……… $\frac{3h}{8}\left(f_{n-3} + 3f_{n-2} + 3f_{n-1} + f_n\right)$

Thus,

$$\int_a^b f(x)dx \cong \frac{3h}{8}\left(f_0 + 3f_1 + 3f_2 + f_3\right) + \cdots + \frac{3h}{8}\left(f_{n-3} + 3f_{n-2} + 3f_{n-1} + f_n\right)$$

$$= \frac{3h}{8}\left(f_0 + 3f_1 + 3f_2 + 2f_3 + 3f_4 + \cdots + 2f_{n-3} + 3f_{n-2} + 3f_{n-1} + f_n\right)$$

Example: Evaluate

$$\int_1^2 x^2 \cos x dx, \quad f(x) = x^2 \cos x \quad \text{with} \quad h = \frac{2\text{-}1}{6} = \frac{1}{6}. \qquad [a,b] = [1,2]$$

True value:

$$\int_1^2 x^2 \cos x dx = \int_1^2 x^2 d \sin x = \left(x^2 \sin x\right)\Big|_1^2 - \int_1^2 \sin x \times 2x dx = 2.7957 + 2\int_1^2 x d \cos x$$

$$= 2.7957 + 2x \cos x \,\big|_1^2 - 2\int_1^2 \cos x dx = 0.0505 - 2 \sin x \,\big|_1^2 = -0.0851$$

Trapezoidal Rule:

$$\int_1^2 x^2 \cos x dx = \frac{h}{2}[0.5403 + 2\times 0.5352 + 2\times 0.4182 + 2\times 0.1592 - 2\times 0.2659 - 2\times 0.8723 - 1.6646]$$

$$= -0.09796$$

Simpson $1/3$ Rule:

$$\int_a^b f(x)dx = \frac{h}{3}[f_0 + 4f_1 + 2f_2 + 4f_3 + 2f_4 + 4f_5 + f_6]$$

$$= \frac{1}{18}[0.5403 + 4\times 0.5352 + 2\times 0.4182 + 4\times 0.1592 - 2\times 0.2659 - 4\times 0.8723 - 1.6646]$$

$$= -0.08507$$

Simpson's $\frac{3}{8}$ rule:

$$\int_a^b f(x)dx = \frac{3h}{8}[f_0 + 3f_1 + 3f_2 + 2f_3 + 3f_4 + 3f_5 + f_6]$$

$$= \frac{3}{48}[0.5403 + 3\times 0.5352 + 3\times 0.4182 + 2\times 0.1592$$

$$- 3\times 0.2659 - 3\times 0.8723 - 1.6646]$$

$$= -0.08502$$

Simpson's $\frac{1}{3}$ rule gives the best result!

In general, Simpson's $\frac{3}{8}$ rule would give the best results.