NUMERICAL SOLUTIONS TO ORDINARY DIFFERENTIAL EQUATIONS

If the equation contains derivatives of an n-th order, it is said to be an n-th order differential equation. For example, a second-order equation describing the oscillation of a weight acted upon by a spring, with resistance motion proportional to the square of the velocity, might be

$$\frac{d^2x}{dt^2} + 4\left(\frac{dx}{dt}\right)^2 + 0.6x = 0$$

where x is the displacement and t is time.

The solution to a differential equation is the function that satisfies the differential equation and that also satisfies certain initial conditions on the function. The analytical methods are limited to a certain special forms of the equations. Elementary courses normally treat only linear equations with constant coefficients.

Numerical methods have no such limitations to only standard forms. We obtain the solution as a tabulation of the values of the function at various values of the independent variable, however, and not as a functional relationship.

Our procedure will be to explore several methods of solving first-order equations, and then to show how these same methods can be applied to systems of simultaneous first-order equations and to higher-order differential equations. We will use the following form

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0$$

for our typical first-order equation.

THE TAYLOR-SERIES MENTOD

The Taylor-series method serves as an introduction to the other techniques we will study although it not strictly a numerical method. Consider the example problem

$$\frac{dy}{dx} = -2x - y, \qquad y(0) = -1, x_0 = 0$$

(This particularly simple example is chosen to illustrate the method so that you can really check the computational work. The analytical solution,

$$y(x) = -3e^{-x} - 2x + 2$$

is obtained immediately by application of standard methods and will be compared with our numerical results to show the error at any step.)

Taylor Series Expansion:

We develop the relation between y and x by finding the coefficients of the Taylor series expanded at x_0

$$y(x) = y(x_0) + y'(x_0)(x - x_0) + \frac{y''(x_0)}{2!}(x - x_0)^2 + \frac{y'''(x_0)}{3!}(x - x_0)^3 + \cdots$$

If we let $x - x_0 = h$, we can write the series as

$$y(x) = y(x_0) + y'(x_0)h + \frac{y''(x_0)}{2!}h^2 + \frac{y'''(x_0)}{3!}h^3 + \cdots$$

Iterative Procedure:

Since $y(x_0)$ is our initial condition, the first term is known from the initial condition $y(x_0) = -1$. We get the coefficient of the second term by substituting $x = 0, y = -1$ in the equation for the first derivative

$$y'(x_0) = \left.\frac{dy}{dx}\right|_{x=x_0} = \left(-2x - y\right)\Big|_{x=x_0} = -2x_0 - y(x_0) = -2(0) - (-1) = 1$$

Similarly, we have

$$y''(x) = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}(-2x - y) = -2 - y'(x) \quad \Rightarrow \quad y''(x_0) = -3$$

$$\Rightarrow \quad y'''(x_0) = 3 \qquad\qquad \Rightarrow \quad y^{(4)}(x_0) = -3$$

We then write our series solution for y, letting $x = h$ be the value at which we wish to determine y:

$$y(h) = -1 + 1.0h - 1.5h^2 + 0.5h^3 - 0.125h^4 + \text{error term}$$

Here shown is a case whose function is so simple that the derivatives of different orders can be obtained easily. However, the differentiation of $f(x,y)$ could be very messy, say, those of $x / (y - x^2)$.

EULER METHOD

As shown previously, the Taylor-series method may be awkward to apply if the derivatives becomes complicated and in this case the error is difficult to determine. In fact, we may only need a few terms of the Taylor series expansion for good accuracy if we make h small enough. The Euler method follows this idea to the extreme for first-order differential equations: it uses only the first two terms of the Taylor series!

Iterative Procedure:

Suppose that we have chosen h small enough that we may truncate after the first-derivative term. Then

$$y(x_0 + h) = y(x_0) + y'(x_0)h + \frac{y''(z)}{2}h^2$$

where we have written the usual form of the error term for the truncated Taylor-series.

The Euler Method Iterative Scheme is given by

$$\begin{cases} y_n' = f(x_n, y_n), \quad y_0 = y(x_0) \\ y_{n+1} = y_n + h \cdot y_n' \end{cases}$$

Example: Using Euler Method with $h = 0.1$, find solution to the following o.d.e.

$$\frac{dy}{dx} = f(x, y) = -2x - y, \qquad y(0) = -1, x_0 = 0$$

$$\Longrightarrow \begin{cases} y_n' = -2x_n - y_n, \quad y_0 = -1, \quad x_0 = 0 \\ y_{n+1} = y_n + 0.1 \cdot \left(-2x_n - y_n\right) = 0.9y_n - 0.2x_n \end{cases}$$

$$y_1 = 0.9y_0 - 0.2x_0 = -0.9 \qquad (-0.9145)$$
$$y_2 = 0.9y_1 - 0.2 \times 0.1 = -0.83 \qquad (-0.8562)$$
$$y_3 = 0.9y_2 - 0.2 \times 0.2 = -0.787 \qquad (-0.8225)$$
$$y_4 = 0.9y_3 - 0.2 \times 0.3 = -0.7683 \qquad (-0.8110)$$

(•) are true values

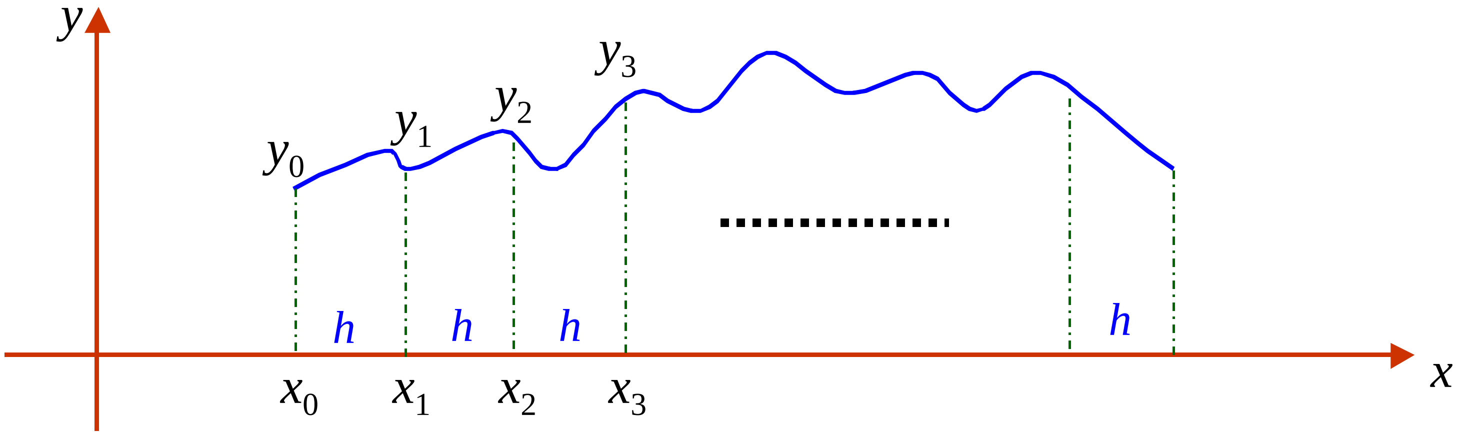
Example (cont.): Let us choose h = 0.001

$$\Longrightarrow \quad \begin{cases} y'_n = -2x_n - y_n, \quad y_0 = -1, \quad x_0 = 0 \\ y_{n+1} = y_n + 0.001 \cdot \left(-2x_n - y_n\right) = 0.999\,y_n - 0.002x_n \end{cases}$$

$$y_1 = 0.999(-1) - 0 = -0.999 \qquad (-0.999002\,)$$

$$y_2 = 0.999(-0.999) - 0.002 \times 0.001 = -0.998003 \qquad (-0.998006\,)$$

Quite accurate, right? What is the price we pay for accuracy? Consider $y(10)$, for $h = 0.1$, we need to compute it in 100 steps. For $h = 0.001$, we will have to calculate it in 10000 steps. No free lunch as usual.



y
x
y_0
y_1
y_2
y_3
h
h
h
h
x_0
x_1
x_2
x_3

THE MODIFIED EULER METHOD

In the Euler method, we use the slope at the beginning of the interval, y'_n to determine the increment to the function. This technique would be correct only if the function were linear. What we need instead is the correct average slope within the interval. This can be approximated by the mean of the slopes at both ends of the interval.

Modified Euler Iteration:

Given an o.d.e.

$$\frac{dy}{dx} = f(x, y) \quad y(x_0) = y_0$$

The modified Euler iteration is:

$$\begin{cases} y'_n = f(x_n, y_n) \qquad y_0 = y(x_0) \\ z_{n+1} = y_n + h y'_n \\ z'_{n+1} = f(x_{n+1}, z_{n+1}) \\ y_{n+1} = y_n + \dfrac{y'_n + z'_{n+1}}{2} h \end{cases}$$

The key idea is to fine-tune y'_n by using $\dfrac{y'_n + z'_{n+1}}{2}$

Example: Solve o.d.e. $\frac{dy}{dx} = -2x - y, \ \ y(0) = -1, x_0 = 0$ **with** $h = 0.1$

Step 1:
$$\begin{cases} y_0' = f(x_0, y_0) = -2x_0 - y_0 = -2\times 0 - (-1) = 1 \\ z_1 = y_0 + hy_0' = (-1) + 0.1\times 1 = -0.9 \\ z_1' = f(x_1, z_1) = -2\times 0.1 - (-0.9) = 0.7 \\ y_1 = y_0 + \frac{y_0' + z_1'}{2}h = -1 + \frac{1+0.7}{2}\times 0.1 = -0.915 \quad (-0.9145) \end{cases}$$

Step 2:
$$\begin{cases} y_1' = f(x_1, y_1) = -2x_1 - y_1 = -2\times 0.1 + 0.915 = 0.715 \\ z_2 = y_1 + hy_1' = -0.915 + 0.1\times 0.715 = -0.8435 \\ z_2' = f(x_2, z_2) = -2x_2 - z_2 = -0.4 + 0.8435 = 0.4435 \\ y_2 = y_1 + \frac{h}{2}(y_1' + z_2') = -0.915 + 0.05(0.715 + 0.4435) = -0.8571 \quad (-0.8562) \end{cases}$$

Step 3:
$$\begin{cases} y_2' = f(x_2, y_2) = -2x_2 - y_2 = -2\times 0.2 + 0.8571 = 0.4571 \\ z_3 = y_2 + hy_2' = -0.8571 + 0.1\times 0.4571 = -0.8114 \\ z_2' = f(x_3, z_3) = -2x_3 - z_3 = -0.6 + 0.8114 = 0.2114 \\ y_3 = y_2 + \frac{h}{2}(y_2' + z_3') = -0.8571 + 0.05(0.4571 + 0.2114) = -0.8237 \quad (-0.8225) \end{cases}$$

THE RUNGE-KUTTA METHODS

The two numerical methods of the last two section, though not very impressive, serve as a good approximation to our next procedures. While we can improve the accuracy of those two methods by taking smaller step sizes, much greater accuracy can be obtained more efficiently by a group of methods named after two German mathematicians, Runge and Kutta. They developed algorithms that solve a differential equation efficiently and yet are the equivalent of a approximating the exact solution by matching the first n terms of The Taylor-series expansion. We will consider only the fourth- and fifth-order Runge-Kutta methods, even though there are higher-order methods. Actually, the modified Euler method of the last section is a second-order Runge-Kutta method.

Fourth-Order Runge-Kutta Method:

Problem: To solve differential equation,

$$\frac{dy}{dx} = f(x, y), \quad y_0 = y(x_0)$$

Algorithm:

$$\begin{cases} y_{n+1} = y_n + \frac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right), \quad y_0 = y(x_0) \quad \textbf{with} \\ \\ k_1 = hf\left(x_n, y_n\right) \\ k_2 = hf\left(x_n + \frac{1}{2}h, y_n + \frac{1}{2}k_1\right) \\ k_3 = hf\left(x_n + \frac{1}{2}h, y_n + \frac{1}{2}k_2\right) \\ k_4 = hf\left(x_n + h, y_n + k_3\right) \end{cases}$$

Proof: 1) Read textbook, or 2) forget about it.

Example: Solve the following o.d.e. using Fourth-Order Runge-Kutta Method

$$\frac{dy}{dx} = -2x - y, \qquad y(0) = -1,\ x_0 = 0,\ h = 0.1$$

Step 1:

$$k_1 = hf(x_0, y_0) = 0.1(-2\times 0 + 1) = 0.1$$

$$k_2 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1\right) = 0.1\left(-2\times\frac{1}{2}\times 0.1 - (-1) - \frac{1}{2}\times 0.1\right) = 0.85\times 0.1 = 0.085$$

$$k_3 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_2\right) = 0.1\times\left(-2\times\frac{1}{2}\times 0.1 + 1 - \frac{1}{2}\times 0.085\right) = 0.08575$$

$$k_4 = hf(x_0 + h, y_0 + k_3) = 0.1\times(-2\times 0.1 + 1 - 0.08575) = 0.071425$$

$$y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = -0.9145125 \quad \boxed{(-0.9145123)}$$

true values

Step 2:

$$k_1 = 0.0715 \quad k_2 = 0.0579 \quad k_3 = 0.0586 \quad k_4 = 0.0456 \quad y_2 = -0.8562 \quad \boxed{(-0.85619)}$$