

CHAPTER 5

THE SIGMA NOTATION AND NUMBER BASES

5A SUMS AND THE SIGMA NOTATION

In the last chapter we found and proved a formula for the sum of a finite number of terms of some sequences. A handy notation for writing sums uses the Greek letter sigma Σ like this:

$$u_1 + u_2 + \dots + u_n = \sum_{r=1}^n u_r$$

We read the right hand side as “the sum of u_r from $r=1$ to $r=n$ ”. The integers 1 and n are known respectively as the lower and upper limits of summation; the variable r is called the index of summation.

Example 5.1. Consider the following sums.

(a) $1+2+3+\dots+n$.

Here, we can put $u_r=r$; then $r=1$ gives the first term in the sum and $r=n$ gives the last term. So we can write

$$1+2+3+\dots+n = \sum_{r=1}^n r$$

(b) $2+5+8+\dots+3n-1$.

In Chapter 4, Example 4.4, we showed that $u_r = 3r-1$, where the first term is given by $r=1$; clearly the last term is given by $r=n$. So we can write

$$2+5+8+\dots+3n-1 = \sum_{r=1}^n (3r-1)$$

(c) $5+15+45+\dots+5(3^{n-1})$

We see here that every term has a factor of 5. So let us start by rewriting the sum as

$$5(1)+5(3)+5(3^2)+\dots+5(3^{n-1}).$$

Now we can see that we can take $u_r=5(3^r)$; the first term is then given by $r=0$ and the last term by $r=n-1$. So we can write

$$5 + 15 + 45 + \dots + 5(3^{n-1}) = \sum_{r=0}^{n-1} 5(3^r)$$

(d) $F_0 + F_2 + F_4 + \dots$

Notice that in this sum the subscripts are all *even*. So we put $u_r=F_{2r}$. The first term is given by $r=0$ and the last by $r=n$. So the sum can be written as

$$F_0 + F_2 + F_4 + \dots + F_{2n} = \sum_{r=0}^{2n} F_{2r}$$

Exercise 5.1. In Example 5.1 (b), we could equally well have put

$u_r = 3r + 2$. In this case, what would the lower and upper limits of summation need to be? Similarly in (c), we could have put $u_r=5(3^{r-1})$. Rewrite the sum in sigma notation with this expression for u_r and the appropriate limits of summation.

Exercise 5.2. Rewrite each of the following sums without using Σ notation.

$$(a) \sum_{r=1}^n 2r \quad (b) \sum_{s=0}^{n-1} 5^s \quad (c) \sum_{i=1}^n (4i-3) \quad (d) \sum_{k=0}^n 3(4^k)$$

Exercise 5.3. What is the value of $\sum_{r=1}^4 u_r$ when $u_r = 5$ for all values of r ? How can we interpret $\sum_{r=1}^n c$ where c is a constant?

The sigma notation is not just a convenient shorthand for writing sums. Most importantly, it gives us a way of working out the sum of a complicated expression by turning it into simpler sums. We can do this by applying some combination of two simple rules.

1. *Take out a common factor.*

$$5(1)+5(3)+5(3^2)+\dots+5(3^{n-1}) = 5(1+3+3^2+\dots+3^{n-1})$$

$$\sum_{r=0}^{n-1} 5(3^r) = 5 \sum_{r=0}^{n-1} 3^r$$

The common factor 5 can be taken outside the sigma sign because it can be taken outside the bracket in the “long hand” version.

2. *Split a sum into two (or more) components.*

$$(1+1^2)+(2+2^2)+(3+3^2)+\dots+(n+n^2) \\ = (1+2+3+\dots+n) + (1^2+2^2+3^2+\dots+n^2)$$

$$\sum_{r=1}^n (r + r^2) = \sum_{r=1}^n r + \sum_{r=1}^n r^2$$

We express these rules in the following theorem.

Theorem 5.1.

$$(a) \quad \sum_{r=m}^n a u_r = a \sum_{r=m}^n u_r$$

$$(b) \quad \sum_{r=m}^n (u_r + w_r) = \sum_{r=m}^n u_r + \sum_{r=m}^n w_r$$

$$(c) \quad \sum_{r=m}^n (u_r - w_r) = \sum_{r=m}^n u_r - \sum_{r=m}^n w_r$$

Proof.

$$(a) \quad \sum_{r=m}^n a u_r = a(u_m + u_{m+1} + u_{m+2} + \dots + u_n) = a \sum_{r=m}^n u_r$$

$$(b) \quad \sum_{r=m}^n (u_r + w_r) = (u_m + w_m) + (u_{m+1} + w_{m+1}) + \dots + (u_n + w_n) \\ = (u_m + u_{m+1} + \dots + u_n) + (w_m + w_{m+1} + \dots + w_n) \\ = \sum_{r=m}^n u_r + \sum_{r=m}^n w_r$$

(c) This can be proved in the same way as (b) and we leave it as an exercise.

An important point to note here is that the limits of the summation are not affected by these manipulations.

By taking out factors and splitting up the sums, we hope to reduce a complicated sum to simpler sums for which we know a formula. From the examples in Chapter 4, we collect together some useful results and write them in sigma notation. We prepared the way for part (a) of Result 5.2 in Exercise 5.2.

Result 5.2. (a) $\sum_{r=1}^n c = nc$

(b) $\sum_{r=1}^n r = n(n+1)/2$ (Example 4.5)

(c) $\sum_{r=1}^n r^2 = n(n+1)(2n+1)/2$ (Example 4.6)

(d) $\sum_{r=0}^{n-1} x^r = \frac{x^n - 1}{x - 1}$, when $x \neq 1$

Proof. Parts (b) and (c) have already been proved in Chapter 4.

(a) In this sum we have $u_r=c$, for $r=1, 2, \dots, n$. So we are adding $c+c+\dots+c$, giving nc altogether.

(d) Let $S=1+x+x^2+\dots+x^{n-1}$, where $x \neq 1$. Multiplying through by x , gives $xS=x+x^2+x^3+\dots+x^n$.

Subtracting, we have $xS-S=x^n-1$. Thus $S(x-1)=x^n-1$, and dividing both sides by $x-1$ gives the required result.

[Note that (d) can also be proved by induction.]

Exercise 5.4. Suppose we altered the lower limit of summation in Result 5.2 (a),(b) and (c) to 0. Would this make a difference to the sum in any of these cases?

Exercise 5.5

Evaluate (a) $\sum_{j=5}^n 2$ (b) $\sum_{r=1}^{100} r^2$

Example 5.1 (Continued).

We now find the formula for the sums in examples (b) and (c).

$$\begin{aligned} \text{(b)} \quad \sum_{r=1}^n (3r-1) &= \sum_{r=1}^n 3r - \sum_{r=1}^n 1, \text{ by Theorem 5.1 (b)} \\ &= 3 \sum_{r=1}^n r - \sum_{r=1}^n 1 \text{ by Theorem 5.1 (a)} \end{aligned}$$

Hence, by result 5.2 (a) and (b)

$$\begin{aligned} \sum_{r=1}^n (3r-1) &= 3n(n+1)/2 - 2 \\ &= n[3(n+1)/2 - 1] \\ &= n(3n+1)/2 \end{aligned}$$

$$\begin{aligned} \text{(c)} \quad \sum_{r=0}^{n-1} 5(3)^r &= 5 \sum_{r=0}^{n-1} 3^r \\ &= 5 \left(\frac{3^n - 1}{3 - 1} \right), \text{ by result 5.2 (d)} \\ &= 5(3^n - 1)/2 \end{aligned}$$

Exercise 5.6. Use Theorem 5.1 and Result 5.2 to find a formula for each of the sums of Exercise 5.2.

$$(a) \sum_{r=1}^n 2r \quad (b) \sum_{s=1}^n 5^s \quad (c) \sum_{i=1}^n (4i-1) \quad (d) \sum_{k=0}^n 3(4^k)$$

Exercise 5.7. Find the formula for $S = 1^2 + 4^2 + 7^2 + \dots + (3n+1)^2$.

5B NUMBER BASES

In this section, we look at different ways of representing a number, first in the familiar decimal notation and then in other number bases, particularly the binary, octal and hexadecimal systems that are used in representing numbers in a digital computer.

The decimal number system, in which we use the ten digits 0,1,...,9 to represent any real number, seems so natural and familiar that it is hard to realize that its use is entirely fortuitous, based presumably on the accident of our having ten fingers to count on. Other peoples and civilizations have based their number systems on other integers, such as 5, 12 or 20. The Babylonians based theirs on 60 and it is to them that we owe the division of both an hour and a degree of turn into 60 minutes and each minute into 60 seconds.

Let us start by reminding ourselves how a string of digits is used to represent a number in base 10. For example, 3589.72 represents the number

$$3(10^3)+5(10^2)+8(10^1)+9(10^0)+7(10^{-1})+2(10^{-2})$$

and in general, the decimal number

$$(a_n a_{n-1} \dots a_0 b_1 b_2 b_3 \dots b_m)_{10}$$

represents

$$a_n(10^n) + a_{n-1}(10^{n-1}) + \dots + a_0(10^0) + b_1(10^{-1}) + b_2(10^{-2}) + \dots + b_m(10^{-m}), \quad \text{where}$$

the a_i 's and b_j 's are each one of the digits 0,1,2,...,9.

We have already remarked in Chapter 1, how the electronic circuitry of a computer easily lends itself to the representation of a two digit number system. For this reason, computers are often designed to use the binary number system, based on the number 2. This system, therefore, is of particular interest to us.

Imitating the representation of a decimal number, but with 2 as base instead of 10, the binary number

$$(a_n a_{n-1} \dots a_0 b_1 b_2 b_3 \dots b_m)_2$$

represents

$$a_n(2^n) + a_{n-1}(2^{n-1}) + \dots + a_0(2^0) + b_1(2^{-1}) + b_2(2^{-2}) + \dots + b_m(2^{-m})$$

where the a_i 's and b_j 's are each one of the digits 0 or 1. The digits to the *left* of the point form the integral part of the number and the digits to the *right* of the point form the fractional part of the number.

Example 5.2.

(a) The binary integer $(1001011)_2$ represents

$$\begin{aligned} 1(2^6)+0(2^5)+0(2^4)+1(2^3)+0(2^2)+1(2^1)+1(2^0) &= 26+23+21+20 \\ &= (75)_{10} \end{aligned}$$

(b) The binary number $(101.11)_2$ represents
 $1(2^2)+0(2^1)+1(2^0)+1(2^{-1})+1(2^{-2}) = 2^2+2^0+2^{-1}+2^{-2} = (5.75)_{10}$

Here, the integral part of $(101.11)_2$ is $(101)_2$ and the fractional part is $(.11)_2$.

Exercise 5.9 (a)

Convert the number $(1010101.00101)_2$ to the decimal system.

(b) To get “the feel” of the binary number system, make a table showing the binary equivalent of each of the decimal numbers 1,2,3,...,16. Allow space for two more rows when you construct your table, so that you can add the octal and hexadecimal equivalents of these numbers.

Exercise 5.10. Suppose the integers x and y are represented in binary by $(a_n a_{n-1} \dots a_0)_2$ and $(a_n a_{n-1} a_{n-2} \dots a_0 1)_2$ respectively. Write down the connection between the values of y and x (that is, express y as a function of x).

Exercise 5.11. The *ternary* number system uses base 3.

(a) Express the ternary number $(1202)_3$ as a sum of powers of 3 and hence find its decimal equivalent.
 (b) Suppose you are given the representation of an integer x in base 3. How you can tell whether x is even or odd? Give an explanation for your answer.

Octal and hexadecimal number systems

Two other number systems commonly used in computer science are the octal system based on 8 and the hexadecimal system based on 16. These bases have the following advantages: they record most numbers using substantially fewer digits than are necessary in base 2; there is a very easy method of converting between binary, which as we have seen is a “natural” language for computers to use, and octal or hexadecimal numbers.

Example 5.3. The octal number $(536.7)_8$ represents the decimal number
 $5(8^2)+3(8^1)+6(8^0)+7(8^{-1}) = (350.875)_{10}$.

Exercise 5.12. Convert the octal numbers (a) $(60.3)_8$; (b) $(2.15)_8$ to the decimal system.

Now let us consider the problem of converting an octal number into a binary number. The following question prepares the ground.

Exercise 5.13. What is the effect on the digits of a decimal number of multiplying it by 10? What is the effect on the digits of a binary number of multiplying it by 2^r ? By 2^{-r} , where $r \in \mathbb{Z}$?

Example 5.3 (continued). We write $(536.7)_8$ as a binary number. First note that since $8 = 2^3$, we can write

$$(536.7)_8 = 5(2^6)+3(2^3)+6(2^0)+7(2^{-3}).$$

We next express each of the digits 5,3,6,7 as a binary number and multiply by the given power of 2. Thus, $5=(101)_2$ and multiplying by 26 moves the digits 6 places to the left, giving $(101\ 000\ 000)_2$. Similarly, $3(2^3)=(11\ 000)_2$, $6(2^0)=(110)_2$ and $7(2^{-3})=(0.111)_2$. Adding these binary numbers, gives

$$(536.7)_8 = (101\ 011\ 110.111)_2,$$

where each block of three bits represents one of the octal digits.

Example 5.4. We write the binary number

$(11011010110.1)_2$ as an octal number by reversing the process described above. We start by splitting the binary string into blocks of three bits, WORKING FROM THE POINT OUTWARDS IN BOTH DIRECTIONS, completing the first (and last block when there are figures after the point), if necessary, by adding extra zeros at the front of the integral part and at the end of the fractional part of the number. We then convert each block into one of the digits 0,1,...,7, giving us a representation of the number in base 8.

Thus, step 1 gives:

$$11011010110.1 = 11\ 011\ 010\ 110.1$$

The first and last blocks are incomplete because they contain only two bits and one bit respectively. So we put one zero on the front of the first block and two on the end of the last block. This gives

$$\begin{aligned} (11011010110.1)_2 &= (011\ 011\ 010\ 110.\ 100)_2 \\ &= 3\ \ 3\ \ 2\ \ 6\ \ .4 \\ &= (3326.4)_8 \end{aligned}$$

In order to express a number in the hexadecimal system we need 16 digits. It is customary to use the digits 0,1,...,9 followed by the letters A,B,C,D,E, and F to represent 10,11,12,13,14 and 15 respectively.

Example 5.5. The number $(5AE7.3)_{16}$ represents the decimal number $5(16^3)+10(16^2)+14(16^1)+7(16^0)+3(16^{-1}) = (22711.1875)_{10}$

Exercise 5.14. Convert the hexadecimal numbers (a) $(A9)_{16}$ (b) $(E.F)_{16}$ to the decimal system. To convert between the binary and hexadecimal systems, we use a similar procedure to the one we used to convert between the binary and octal systems. This time, however, we split the binary string into groups of four bits instead of three.

$$\begin{aligned} \text{Example 5.6 (a)} \quad (B3.5)_{16} &= (1011\ 0011.0101)_2 \\ \text{(b)} \quad (101011101.11)_2 &= 0001\ 0101\ 1101.1100 \\ &= 1\ \ 5\ \ D\ \ .\ C \\ &= (15D.C)_{16} \end{aligned}$$

Exercise 5.15.

Write the following binary integers in both the octal and the hexadecimal system (i) $(1011101)_2$; (ii) $(1011101011)_2$. Suppose the binary representation of an integer has n bits. Give an expression for the number of digits required to represent the integer in base 8 and in base 16.

Converting decimal integers into other bases

Suppose that we want to convert a decimal integer x into binary. Then we require to find bits a_0, a_1, \dots, a_n such that

$$x = a_n(2^n) + a_{n-1}(2^{n-1}) + \dots + a_2(2^2) + a_1(2) + a_0(1).$$

If we *divide* x by 2, we will obtain a quotient: $q = a_n(2^{n-1}) + a_{n-1}(2^{n-2}) + \dots + a_2(2) + a_1(1)$ and a remainder: $r = a_0$. Thus we can discover the value of a_0 by dividing x by 2 and noting the remainder.

Now, if we repeat this process with q in place of x , we divide q by 2 and obtain a new quotient

$$q' = a_n(2^{n-2}) + a_{n-1}(2^{n-3}) + \dots + a_2(1)$$

and a remainder: $r' = a_1$.

Repeating this process, we can obtain successively the values of a_2, a_3, \dots, a_n .

We illustrate this process with an example. Suppose we want to express the decimal integer 25 in binary. We divide 25 by 2, noting separately the quotient and the remainder. Then we divide the quotient by 2 and again note the remainder. We repeat this step until we reach a zero quotient. Then we stop. We can set out the division as follows.

Step	quotient	remainder
0	2 25	
1	2 12	1 = a ₀
2	2 16	0 = a ₁
3	2 3	0 = a ₂
4	2 1	1 = a ₃
5	0	1 = a ₄

Fig 5.1.

Thus we have $25 = (11001)_2$. You can easily verify that this is correct by converting the binary integer back into decimal. Note carefully that the division process gives the digits in the binary representation IN REVERSE ORDER as the successive remainders.

Example 5.7. We convert $(75)_{10}$ to binary by this method. step quotient remainder

Step	quotient	remainder
1	37	1
2	18	1
3	9	0
4	4	1
5	2	0
6	1	0
7	0	1

Recording the remainders in reverse order, gives $(75)_{10} = (1001011)_2$ (see Example 5.2, where we did this conversion the other way round).

The same procedure, except that we divide by b at each stage instead of by 2, can be used to express a decimal integer in any other base b , where $b \geq 2$. Here is the procedure in the form of an algorithm.

Algorithm 5.1. To convert an integer x in base 10 to an integer in base b .

- Step 0a. Input a positive integer x .
- Step 0b. Input an integer $b \geq 2$.
- Step 1a. $a := x$.
- Step 1b. $k := 0$.
- Step 2. While $a > 0$, do
 - Step 2a. Divide a by b .
 - Step 2b. $q_k := \lfloor a/b \rfloor$ the quotient in Step 2a.
 - Step 2c. $r_k := a - bq_k$
 - Step 2d. $a := q_k$
 - Step 2e. $k := k + 1$.
- Step 3. Output $(r_k r_{k-1} \dots r_0)_b$
- Step 4. Stop.

Exercise 5.16. Work through this algorithm for $x=13$ and $b=2$, recording

the values of q_k, r_k, a, k at each iteration of the loop.

Exercise 5.17. Use Algorithm 5.1 to convert the decimal integer 117 into (a) the binary (b) the ternary number systems.

Exercise 5.18. How can we express a repeating “decimal” in the binary system as a binary fraction? The first step is to think how we might express a repeating decimal in base 10 as a fraction, and then try a similar method in base 2. In case you are not familiar with the method in base 10, here is an explanation and example. Suppose the number x has an infinite decimal expansion with a repeated block of r digits. We can find an expression for x as a fraction by multiplying x by 10^r and subtracting x from the result. The following example illustrates the method.

Example. Suppose $x = 0.534\ 534\ 534\dots$ (A)
There are 3 digits in the repeated block, so we multiply both sides by 10^3 . This gives
 $1000x = 534.534\ 534\dots$ (B)
Subtracting (A) from (B) gives
 $999x = 534.000\dots$
Thus $x = 534/999 = 178/333$.

(a) Suppose the decimal number x is represented in base 2 by the infinite binary sequence $(0.1111\dots)_2$. (Notice there is just one bit in the repeated block.) Write down the binary representation of $2x$ and subtract x . Hence prove that $x = 1$.

(b) Find a decimal fraction that has the infinite binary representation
(i) $(0.101010\dots)_2$; (ii) $(0.110110110\dots)_2$.