CHAPTER 1

FUNDAMENTALS

The topics that will be covered in this chapter include an introduction to a basic counting technique and the language of sets and subsets. These topics are fundamental to the study of Discrete Mathematics and Computer Science. Indeed, the language of sets is fundamental to all mathematical study and you will use the ideas in this first chapter in many of the area of Computer Science. So you are advised to go through it very carefully.

We shall continue the study of sets in Chapter 2 and return to the use of the multiplication principle of counting in Chapter 7; other counting techniques will be introduced in Chapters 2 and 3.

1A A COUNTING PRINCIPLE

We shall start by considering three related counting problems. You should try to solve these for yourself before reading further.

Exercise 1.1

You wish to visit a friend, who lives on the other side of the city, by public transport. As there is no direct service between your homes, the journey has to be completed in two stages. From your home to the centre, you have three alternatives, two different bus routes (let us call them bl and b2) and a rail route (r). From the centre to your friend's house, you can take either a further bus route (b3) or an underground train (u). In how many different ways can you make the complete journey?

You can easily solve this problem by enumerating the possibilities (i.e. making a list and checking that you haven't left any out). But we ask you in addition to think of a way of *illustrating* the solution that makes the reason for your answer clear.

Exercise 1.2 From the menu for a hotel meal, you may select one of five starters, one of four main dishes and one of six deserts. How many different meals comprising a starter, a main course and a desert is it possible to order?

Your solution to Ex. 1. 1 may suggest a *method* **by** which you can solve this problem without enumerating all the possibilities.

Exercise 1.3 You toss a coin six times in succession; when it comes down showing "heads", you record a 1 and when it comes down showing "tails", you record a 0. How many different sequences of O's and/or l's is it possible to obtain?

Tree diagrams

One way of illustrating the solution to Ex. 1. 1 is by means of a **tree diagram,** as shown in Fig.1.1.

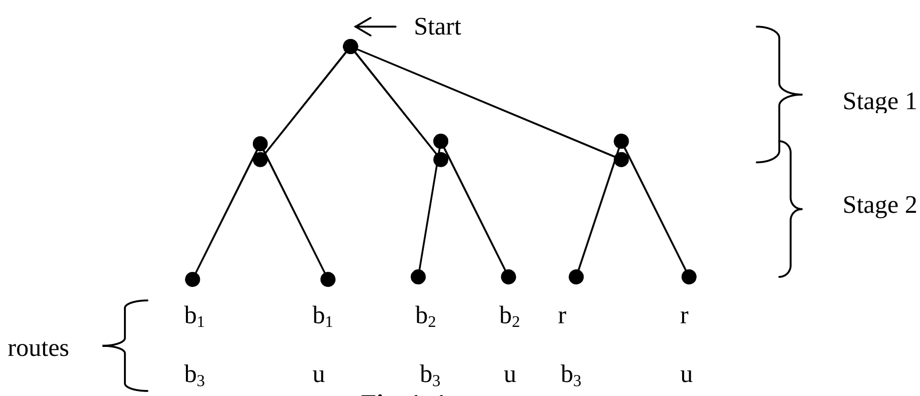

Fig. 1. 1.

Notice that each of the routes corresponds to a **path** in the tree, from the **start vertex** at the top of the tree to an **end vertex** on the lowest level of the tree. This illustrates the fact that as each of the three choices you can make at *stage* 1 can be followed by any one of the two choices that you can make at *stage* 2, you have (3) (2)=6 ways of completing the journey.

The problem described in Ex.1.2 can also be divided into stages. Stage 1: select one of the five starters; *stage* 2: select one of the four main dishes; So each of the 5 choices at *stage I* is followed by any one of 4 choices at *stage* 2, giving (5)(4)=20 different combinations of starter and main dish. But each of these 20 combinations can be followed by any one of 6 choices for desert at *stage* 3, giving a total of (5)(4)(6)=120 different meals.

This approach gives the following useful counting principle.

The Multiplication Principle. *Suppose that a process can be completed in* **k** *stages. If there are* m_1 *choices at stage 1 and each of these can be followed* by m_2 *choices at stage 2, . .. and finally each of the choices at stage* **k-1** *can be followed* **by** m_k *choices at stage k, then the number* **of** *ways in which the process can be completed is given* by *the product* $m_1 m_2 \ldots m_k$.

Exercise 1.4.

The College Computing Society is electing a new committee. There are 5 nominations for the office of President, 2 nominations for Secretary and 3 for Treasurer. How many different committees would it be possible to elect?

Now let us turn to the solution of Ex.1.3. We need to calculate the number of possible sequences of six symbols, each of which is either a 1 or a 0. We can regard this as a 6-stage problem in which we have just two choices at *stage* 1 and each of these is followed **by** two choices at *stage* 2, and so on (we leave you to draw the tree diagram . to illustrate this). Thus we may apply the ***Multiplication*** *Principle* with $m_1 = rn_2 = \ldots . m_6 = 2$. Hence the number of ways of completing the 6 stages is (2)(2)(2)(2)(2)(2)=26, which gives the total number of different sequences as 64.

Binary strings

There are many practical problems in which there are just two possibilities at each stage, or where we want to sort a list of items into two categories, such as the "heads" and "tails" recorded in Ex.1.3. For example, when a factory made item is tested to see whether it meets the specifications, it will be either "acceptable" or "unacceptable"; an animal can be "male" or "female"; a light can be ..on" or .,off"; some questionnaires are designed to have the answer "Yes" or "No" to each question. In all these situations, one of the categories can be coded by the symbol 1 and the other by the symbol 0. Now a computer is designed to recognize two states, such as "high" voltage and " low" voltage. It can therefore handle and store information presented in the form of a sequences of zeros and ones.

A finite sequence of zeros and ones is known as a **binary string.** Each digit (0 or 1) in the string is called a **bit** and a binary string of n bits is called a **n-bit binary string.** Thus 01 is a 2-bit binary string and 10110 is a 5-bit binary string; the complete list of solutions to Ex.1.3 gives all the 6-bit binary strings. Generalizing the method by which we showed that there are exactly 2^6 6-bit binary strings, we can prove the following general result.

Theorem 1.1 *There are 2n n-bit binary strings.*

Proof. We can find any n-bit binary string in n stages, where for i=1,2,...,n, stage i consists of choosing the i th digit in the sequence. At each stage, we have exactly 2 choices, regardless of which

choice we made at the previous stage. Thus we may apply the Multiplication Principle with $m_1=m_2=...$ $=m_n=2$, giving 2^n n-bit binary
strings altogether.

Exercise 1.5.
(a) Draw a tree diagram to illustrate all possible 4-bit binary strings.
(b) In a family with four children, how many different sequences of sons and daughters would it be possible to have?

Exercise 1.6.

In a questionnaire, there is a list of 15 questions, each of which must be answered by ringing one of the words "Yes" or "No". How many different sequences of answers are possible, assuming that every question is answered?
Now suppose the questionnaire is modified to allow one of three responses, "Yes", "No" or "Don't know" to each question. How many sequences of answers are now possible, if every question is answered?

Exercise 1.7. Suppose you toss a die three times, noting the number thrown at each toss, so that you end up with a sequence of three numbers. How many possible sequences can result?

1B SETS AND SUBSETS

By a **set** we simply mean a collection or class of objects. The objects in the set are called its **members** or **elements.** We usually use an upper case letter to denote a set and a lower case letter to denote a member of the set. We write $y \in X$ to denote that (the element) y belongs to (the set) *X* and $y \notin X$ to indicate that y is not a member of the set X.

Specifying sets
To specify a set, we must describe its members in an unambiguous way. One way of doing this is to list the members of the set, enclosing the list in a pair of brace brackets.

Example 1. 1
The set D of' digits that we use in the decimal number system is D={0,1,2,3,4,5,6,7,8,9Y; the set of bits is B={0,1Y. We can say that $S \in$ D, but $S \notin$ B.

Example 1. 2
Let H be the set of integers from 1 to 100, Now, here we cheat a bit and write H={1,2,3,...,100}. The three dots
between the commas mean "et cetera" and we can use them provided our meaning is clear. In a similar way, we
would take the set K={1,3,5,7.... } to mean the set of all positive odd numbers.

It is convenient to denote certain key sets of numbers by a standard letter. We shall denote the set {0,1,~1,2,-2,3,-3,. of integers by Z. Note that Z includes zero and the negative whole numbers as well as the positive ones. We write N={0,1,2,3.... } for the set of **natural** or counting numbers and Z^+={1,2,3.... } for the set of **positive integers.** We denote the set of **real numbers** by R.

Another way of specifying a set is by giving **rules of inclusion** that distinguish members of the set from objects not in the set. The context of the problem in which the set arises determines an underlying set, which we call the **universal set** for the problem, from which our elements will be drawn. So, for example, if our subject is a set of leopards, the universal set, explicitly stated or implied by the context, might be all wild animals in Africa or all animals in London Zoo or all animals belonging to the cat family.

Example 1.3

To specify the set H of Example 1.2, we could write

$H = \{n: n \in Z$ and $1 \leq n \leq 100\}$

This tells us that the *universal set* is Z and the other- *rule of inclusion* is that n must be between 1 and 100 inclusive; the colon stands for the words "such that", so we could read this as "H is the set of integers n such that $1 \leq n \leq 100$". Similarly, we could specify the set K of Example **1.2** as $K = \{m: m \in Z^+$ and m is odd$\}$.

Example 1.4

Let X be the set of real numbers that satisfy the equation $X^2-3x+2=0$. Then we could write $X = \{x: x \in R$ and $x^2-3x+2=0\}$ and read this as "X is the set of numbers x such that x is real number and $x^2-3x+2=0$".

Exercise 1.8

Specify the set X of Example 1.4 by the listing method. An even integer is a number that can be expressed in the form 2m, where $m \in Z$. We can write the set {0,2,-2,4,-4.... } of even integers by the *rules* **of** *inclusion* method as $\{a: a=2m$ and $m \in Z\}$ or more simply, as $\{2m: m \in Z\}$. Notice that even integers can be negative, as well as positive, and that 0 is an even integer. Any odd integer can be obtained by adding 1 to (or subtracting 1 from) some even integer. Thus the set {1,-1,3,-3,5,-5.... } of odd integers can be expressed as $\{2m+l: m \in Z\}$ or $\{2m-l: m \in Z\}$

Example 1. 5

The set {. . . -20, -10,0,10,20, . . . } can be expressed by the *rules of inclusion method as* $\{10a: a \in Z\}$ and the set {1,10,100,1000.... } as $\{10^r: r \in N\}$.

Exercise 1.9.

Describe the following sets by listing their elements
(using dots where necessary).

(a) $\{x: x \in Z$ and $-1 \leq x \leq 4\}$ (b) $\{r: r \in N$ and $r \leq 4\}$
(c) $\{3b-l: b \in Z, -3 \leq b \leq 2\}$ (d) $\{3^r: r \in Z\}$

Exercise 1.10.

Describe the following sets by giving a universal set and appropriate rules of inclusion.
(a) {12,13,14,15,16,17} (b) {0,5,-5,10,-10,15,-15...}
(c) {-1,-2,-3,...} (d) {1,2,4,8,16,32.... }

Subsets

Given two sets A and B, the set *A* is said to be a **subset** of B if every element of *A* is also an element of B. When this is the case, we write $A \subseteq B$. Notice that $A \subseteq A$, for all sets *A.* If it is true that both AGB and BGA, then A and B must contain the same elements. In this case, we say that the sets A and B are **equal** and write $A=B$.

The set containing no elements is known as the **empty** or **null set** and usually denoted by the symbol ϕ (read "null set"). We regard ϕ as a subset of every set.

Example 1.6 $\mathbf{Z^+}$ is a subset of N and N is a subset of Z. We can express this by writing $Z^+ \subseteq N \subseteq Z$.

Example 1.7 Suppose $X=\{x: \in R$ and $x^2-3x+2=0\}$, Y={1,2} and Z={1,1,2,1,2}. Then since each of these sets contain just the numbers 1 and 2, we have X=Y=Z. These equalities illustrate the fact that a set is *determined by its elements*, the method of specification is not important; further, we may ignore repetitions of elements and also the order in which the elements are written.

Size of a set

We call a set **finite** when it contains a finite number of elements and otherwise it is called **infinite.** The number of distinct elements in a finite set S is called its **size or cardinality,** and is written IS], which we read "size of S". (Notice that we use the word distinct in mathematics to mean "distinguishable" or "different".) Thus the sets

X,Y,Z of Example 1.7 each have size 2, the set *H={n:* $n \in Z$ and $1 \leq n \leq 100$ has size 100 whereas N, Z, and R are all examples of infinite sets. We write $|X| = 2$, $|H| = 100$. We define $|\phi| = 0$.

Exercise 1.11.

Which of the following sets are equal? Give the cardinality of each.
(a) *{x,y,z} (b) {z,x,y} (c) {x,y,x,y,z} (d) {y,z,z,x}*

Exercise 1.12. Fried bacon, eggs and tomatoes are on the breakfast menu at your hotel. You may, of course, not wish to order any of these, but if you do, you may choose any one, two or three of them. How many different possibilities arise?

You should have found 8 possibilities. Now, we can code each of the possible choices for your order using a 3-bit binary string. We record a "1" if you choose an item and a "O" if you reject it. The first bit represents your decision on bacon, the second on eggs and the third on tomatoes. But each possible order can also be regarded as a subset of the set F = {b,e,t}, (where b = bacon, *e* = eggs and t = tomatoes). The correspondence between the subsets and the 3-bit codes is given in the table below.

Subset	0	*{b}*	*{e}*	{t}	*{b,e)*	*{b,t}*	*{e,t}*	*{b,e,t}*
code	000	100	010	001	110	101	011	111

In general, suppose that that the n elements of a finite set S have been listed in a fixed order, so that S={x_l, x_2,.-…,x_n}, say. Then given any subset A of S, we can decide for each element $x_i \in S$ whether x_i is in the subset A or not. Thus corresponding to each subset A *of* S, we can construct an n-bit binary string such that the ith bit, counting from the left, is 1 if $x_i \in A$ and is 0 if $x_i \in A$**.** The n-bit string determined by the subset A using this rule is called the **bit vector** of A. Thus the codes in the table above are each the **bit vector** of the corresponding subset.

Example 1. 8
For the set D={0,1,2,3,4,5,6,7,8,9} of Example 1.1, the subset of digits represented by the bit vector 1100010101 is {0,1,5,7,9} and the subset {0,1,2,3,4} is represented by the bit vector 1111100000.

Exercise 1.13.
For the set *V* = {a,e,i,o,u}, give the bit vector for the subsets (a) *{a,i,o};* (b) {e}; (c) *V;* (d) ϕ. Which subset is represented by the bit vector 10001?
How many subsets are there of a set containing 6 elements? Every subset of a set containing 6 elements can be represented in a unique way as a 6-bit binary string. Conversely, every 6-bit binary string represents a unique subset. So there are the same number of subsets as there are 6-bit binary strings. But we showed in the solution to Ex.1.3 that this number is 26 . Generalizing this idea gives the following theorem.

Theorem 1.2 *A set with a finite number n of elements has exactly 2^n subsets.*

Proof. Let S = $\{x_1, x_2 ......x_n\}$**.** Then every subset of S can be uniquely represented by an n-bit binary string and conversely. But by Theorem1.1, there are exactly 2^n, n-bit binary strings. Hence there are just 2^n subsets of S.

We call the set of all subsets of a set S the *power set of* S, denoted by 91(S). Notice that both ϕ and S are elements of $\wp(S)$*;* they correspond respectively to the bit vectors in which each bit is 0 and in which each bit is 1.

The situation in which we have a set whose elements are themselves sets
calls for some care in the use of notation.

Example 1.9 Let S={a,b,c}.
Then *P(S)={* ϕ *,{a},{b},{c},{a,b},{a,c},{b,c},S}.*

	We write	*We read*
	$a \in S$	a is an *element* of the set S
	$\{a\} \subseteq S$	{a} is a *subset of* S
But	$\{a\} \in P(S)$	{a} is an *element of* P(S)

Exercise 1.14

Let S = {1,2,3}. Which of the following statements are correct?
(a) $1 \in S$ (b) $\{1,2\} \in S$ (c) $\phi \subseteq S$ (d) $\{3\} \subseteq S$ (e) $\phi \in S$
(f) $\{3\} \subseteq P(S)$ (g) $\phi \in P(S)$ (h) $\{3,1\} \in P(S)$ (i) $2 \in P(S)$.

Cartesian products
Look again at Fig.1.1 which illustrates the solution to Ex.1.1. You will see that at the foot of the diagram we have listed the possible routes in the form of a table. An equivalent way of listing the routes would be to write each in the form of an **ordered pair,** as we would write the pair of coordinates of a point in the cartesian plane. So, for example, the ordered pair (b_2,u) represents the journey using the bus route b_2 for the *first* stage and the underground for the *second* stage. Let $A = \{b_l$, b_2 , r} and B = {b_3 , u} be the set of routes available for the first and second stages of the journey respectively. 'Then the set

$$\{(b_l\, b_3),(b_l\, u),(b_2,b_3),(b_2,u),(r,b_3),(r,u)\},$$

of all possible routes expressed as ordered pairs, is called the
cartesian product of A and B. It is written A *x* B and read *"A cross B".*

In general, given any two sets X and Y, then the *cartesian product* of *X*
and Y is the set $X \times Y = \{(x,y): x \in X, y \in Y\}$.

Exercise 1.15.

List the elements of the sets *A x* B and *B x A,* where A = {0,1} and B = {1,2,3}. Find a non-empty set

C such that *AxC = CxA.*
Exercise 1.16.
Suppose that A and B are finite sets with]AI = *m* and
IBI = n. Prove that *]A X BI = mn. [Hint: What are the elements of AxB?*
Use the Multiplication Principle to count them.]

Example 1.10 In Ex.1.2, we could represent your choice of starter, main dish and desert by an *ordered triple, (s,m,d), where* $s \in S$, the set of starters, $m \in M$, the set of main dishes and d **E** D, the set of deserts. The cartesian product

S *x M x D = {(s,m,d):* s $\in$ *S,* m $\in$*M, d* $\in$ *D}* gives the set of all possible combinations from the menu. Notice that in Ex.1.2, we calculated the *size* of the set S *x M x D.*

Example 1.11 We are used to writing pairs of **cartesian coordinates** *(x,y)* to describe the position of any point in the plane, or 2-dimensional space. The set {(x,y):x$\in$R,y$\in$R} of all such coordinates is the cartesian product set R x R. Similarly, the set {(x,y,z):x,y,z $\in$R} of all three dimensional cartesian coordinates, is the cartesian product set R x R x R. It is from these sets that the name "cartesian product" is derived.

We can generalize the idea of an ordered pair and an ordered triple to an **ordered n-tuple,** $\{x_1, x_2, x_3....x_n\}$ **,** and the cartesian product of two or three sets to the cartesian product of n sets, for any n $\in Z^+$.

When each of the sets in a cartesian product is the same, as for example in the case of the sets R x R and R x R x R, we often use an abbreviated notation. We denote R x R by R^2 and R x R x R by R^3 and, in general, we denote the set *A x A x... x A, by* A", where there are n A's altogether in the product

Exercise 1.17.
Let S = {0,1,2}. What is the cardinality of S^2? How many elements of S3 have a zero in the first position? How many elements of S^4 have a zero in both the first and fourth positions?

Example 1. 12
An n-bit binary string is an example of an ordered n-tuple, even though we write it without the brackets and without commas between the entries. Let B={0,1Y (the set of bits). Then the set of all n-bit binary strings is

$$\{a_1a_2... a_n,:a_1 \in B,a_2 \in B,...,a_n \in B\}$$

and this set can be denoted by B^n.