CHAPTER 8

LOGIC

8A SYMBOLIC STATEMENTS AND TRUTH TABLES

Logical argument and deductive reasoning are central to mathematics; and we could not write or test the validity of a computer program without them. In these next two sections we look at the symbolic representation of statements and the laws of logic.

Propositions

The statements with which we are concerned are known as propositions. These are statements that are either true or false. Statements that could be considered true by one observer but simultaneously considered false by another observer are not propositions.

Example 8.1.

The following sentences are propositions.

(a) This animal is a cat.
(b) This polygon has 4 sides.
(c) The positive integer *n* is prime.
(d) The real number x is greater than 5. The following sentences are not propositions.
(e) Are you coming to the Disco?
(f) Hurry up, then!
(g) My bag is heavy.

The statements (a) to (d) are either true of false, depending on the circumstances. The important thing to understand is that any two observers would agree about whether each of these statements is true or false in the same circumstances. The sentence (e) is a question and (f) is a command: these cannot be either true or false and so are not propositions. The sentence (g) is not a proposition because one observer might consider it true and another consider it false.

We shall denote propositions by lower case letters, such as p and q. We give each a "truth" value, either 1 for true or 0 for false. We can also define a truth set for the proposition. The truth set *P* for the proposition p contains all the circumstances under which p is true; its complement *P'* contains all the circumstances under which p is false.

Example 8.2. Suppose we toss a coin three times. Let p denote the proposition "The first toss is a head", and q denote the proposition "The third toss is a tail". The set of all possible outcomes (or results) of this experiment is U ={HHH,HHT,HTH,HTT,THH,THT,TTH,TTT}. The truth set for p is

P={HHH,HHT,HTH,HTT} and the truth set for q is Q={HHT,HTT,THT, TTT}.

Exercise 8.1. Suppose I toss a coin three times (see Example 8.2).
Find the truth set for each of the following propositions.
(a) I get a head more often than a tail.
(b) The result of each toss is the same.
(c) The result of the first and last toss is different.

Propositions that are always true are called **tautologies** and those that are always false are called **contradictions**.

Example 8.3. The following propositions are tautologies.
(a) The result of tossing a coin is either a head or a tail.
(b) The members of a set are called its elements.
(c) $(x+1)^2 = x^2+2x+1$.
The following propositions are contradictions.
(d) The value of x is less than 3 and greater than 10.
(e) $x = x+1$.

Proposition (b) is really just a definition of the term *element;* all definitions are tautologies. Proposition (c) is an example of an algebraic identity. An **identity** is true for all values of x, whereas an *equation* in x is true only for some special value(s) of x (those that are said to satisfy the equation) and false for all other values of x. All identities are tautologies. There are no values of x for which propositions (d) and (e) are true; the truth set of each is therefore the empty set *0.*

Exercise 8.2.

Which of the following sentences are *propositions?* Say
also whether any of the propositions is a tautology or a contradiction.
(a) The chemical formula for water is 1120.
(b) This subject is easy.
Cc) A kilo of lead weighs more than a kilo of feathers.
(d) The binary representation of x has exactly three bits.
(e) The binary representation of 5 is (101)2.
(f) Call "heads" or "tails".

The negation of a proposition p, is the proposition that is true when p is false and false when p is true. We denote the negation of p by "~ p, read *"not p".* The truth set for '-p is the complement of the truth set for p.

Example 8.4.

(a) Let p denote the proposition "The integer *n* is prime"; then "-p is the proposition "The integer *n* is *not* prime".
(b) Let p denote the proposition "x = x +l". Then '-p is the proposition "x ≠ _x+l". Notice that the negation of the contradiction p is a tautology.

Similarly, the negation of a tautology will always be a contradiction.

Compound statements

We can join two or more propositions together by such words as "and", "or", "if.. .then", to form compound statements.

Example 8.2 (continued)

We combine the propositions p and q to give the following compound statements.
(a) "The first toss is a head and the third toss is a tail" is denoted symbolically by $p \wedge q$.
(b) "The first toss is a head or the third toss is a tail" is denoted symbolically by *pvq.*
(c) "The first toss is a head or the third toss is a tail, but not both" is denoted symbolically by $p \oplus q$.

Notice that or in mathematics is used in its inclusive sense, unless we specify otherwise, so that (b) includes the possibility that the first toss is a head and the third toss is a tail, whereas (c) does not. The symbol ~ is known as exclusive or.

We have already found above the truth sets for the propositions p and q or Example 8.2. We now find the truth set for each of the compound statements (a), (b) and (c).

(a) The truth set for $p \wedge q$ is {HHT,HTT} = *P n* Q.
(b) The truth set for pvq is {HHH,HHT,HTH,HTT,THT,TTT} = *P u* Q.
(c) The truth set for $p \oplus q$ is {HHH,HTH,THT,TTT} = $P \oplus Q$.

Exercise 8.3.

Let p denote the proposition "$x > 10$" and q denote the proposition "$x < 100$". Write the following statements in symbolic form.

(a) $x \geq 100$; (b) $10 < x < 100$; Cc) $x \leq 10$ or $x \geq 100$;
(d) Either $x \leq 10$ or $x \geq 100$, but not both.

Exercise 8.4.

Suppose *n* is a positive integer. Let p denote the proposition *'n* 20" and q denote the proposition *"n* is a prime". Express in words, as simply as you can, the following statements as conditions on *n.*

(a) ~ p (b) p Λ q (c) ~(~q) (d) *(~p) v q.*

We may determine the truth values of each of the compound statements *pvq,* pΛq, $p \oplus q$, from the truth values of their constituent propositions, *p* and q, by considering each combination of p true or false with q true of false. This is most easily done in the form of a truth table, as shown below.

p	q	pΛq	pVq	p⊕q
0	0	0	0	0
0	1	0	1	1
1	0	0	1	1
1	1	1	1	0

Fig.8. 1.

We see that each of the compound statements is logically distinct, because it has its own distinct pattern of 0's and l's. Further, this pattern is identical to the pattern of 0's and l's in the membership tables of *P nQ* in the case of p Λ q, of *P u Q* in the case of p v q and of *P*⊕ *Q* in the case of p⊕q.

When two statements p and q have the *same* truth table, they are called logically equivalent and we write p=q.

Result 8.1. ~(~p) = p.

Proof: To prove this result, we construct the truth table for ~(~p) and compare it with the truth values of p.

p	~p	~(~p)
0	1	0
1	0	1

Fig.8.2.

We see that whatever the truth value of p, the truth value of ~(~p) is the same. this proves that p = *~(~p)*. .

Example 8.5. Let p denote the proposition "x is greater than 5". Then ~ *p* denotes the proposition "x is not greater than 5", and ~ (~p) denotes the proposition "It is not true that x is not greater than 5". This means the same as " x is greater than 5", but the double negative makes the sentence awkward and its meaning less obvious. Results 8.1 says that we can always avoid double negatives and, in the interests of clearly, we should do so.

Example 8.6: We use a truth table to show that ~(p Λ (~q)) = (~p) v q

p	q	-q	pΛ(~q)	—(pΛ(-q))	~p	(~p)vq
0	0	1	0	1	1	1
0	1	0	0	1	1	1
1	0	1	1	0	0	0
1	1	0	0	1	0	1

Fig 8.3.

Exercise 8.5. Construct the truth tables for (p v q) and (~-p)Λ(~ q) and hence show that
~(p v q) = (~p)Λ(~q). Express both sides of this equation

All tautologies are logically equivalent because every tautology has only the truth value 1. Similarly, all contradictions are logically equivalent, since each takes only the truth value 0. We can therefore use the one symbol, *T* to denote any tautology and the symbol *F* to denote any contradiction.

Example 8.7. The truth table for *pvT* is shown in Fig.8.4.

p	T	pv T
0	1	1
1	1	1

Fig 8.4

The table shows that *pvT* = *T*. In practice that means that the compound statement (proposition p) or (tautology) is always true, whatever the truth value of p.

Exercise 8.6.

Construct a truth table for each of the following compound statements and hence find simpler propositions to which each is equivalent.

(a) *pvF* (b) pΛT

8B THE CONDITIONAL CONNECTIVES

In mathematics and when writing computer programs, as well as in everyday language, we often use the word “if” to connect two statements. However, “if” can be used in more than one way, as the following examples illustrate.

(a) *If $n=9$, then* $\sqrt{n}$ is an integer.
(b) I go to the Bridge Club *only if* today is Tuesday.
(c) The angles of a triangle are equal *if and only if* the triangle is equilateral.

Without altering the meaning, we can rewrite each of these statements using the word "implies".

(a) *$n=9$, implies* $\sqrt{n}$ is an integer.
(b) I go to the Bridge Club *implies* today is Tuesday.
(c) The angles of a triangle are equal *implies* the triangle is equilateral *and* the triangle is equilateral *implies* the angles of the triangle are equal.

The compound statement "p implies q" is written symbolically as $p \rightarrow q$. As we have seen, this statement can be read in other ways. It has the same meaning as any of the following.

If p, then q;
If $n=9$, then $\sqrt{n}$ is an integer.

q **if** *p*
$\sqrt{n}$ is an integer **if** $n=9$.

p **is a sufficient condition for** q;
$n=9$ is a sufficient condition for $\sqrt{n}$ to be an integer.

p **only if** q;
n=9 **only if** $\sqrt{n}$ is an integer.

q **is a necessary condition for** p;
$\sqrt{n}$ **is an integer is a necessary condition for** *n* to be 9.

We write the compound statement "$p \rightarrow q$ and $q \rightarrow p$", as $p \leftrightarrow q$. This has the same meaning as each of the following.

p **if and only if** q;
The angles of a triangle are equal **if and only if** the triangle is equilateral.

p **is a necessary and sufficient condition for** q.
That the angles of a triangle are equal **is a necessary and sufficient condition for** the triangle to be equilateral.

Exercise 8.7. Suppose that *m* and *n* are positive integers. For each of the following pairs of propositions p and q, which of the compound statements $p \rightarrow q$, $q \rightarrow p$, $p \leftrightarrow q$, are true?

(a) p: m+n10; q: m=3 and $n=7$.
(b) p: $m > 2$; q: $m^2 \geq 9$.
(c) p: *m* and *n* are both odd; q: $m + n$ is even.
(d) p: The square root of *n* is irrational; q: n=2.

Exercise 8.8. For each of the pairs of propositions p and q in Exercise 8.7, which of the following is true?

p is a necessary condition for q;
p is a sufficient condition for q;
p is a necessary and sufficient condition for q.

Exercise 8.9. Let *p, q, r* denote the following propositions.

p: "The triangle is isosceles"; q: "The triangle is equilateral"; r: "The triangle has a pair of equal angles".

Which of the following propositions are true for all triangles? (a) p-~q (b) $r \rightarrow p$ Cc) $p \leftrightarrow r$ (d) $(\sim p) \rightarrow (\sim q)$ (e) $(\sim q) \rightarrow p$ (f) $(\sim p) \rightarrow (\sim r)$.

Truth tables for $p \rightarrow q$ and $p \leftrightarrow q$

We consider the truth value of $p \rightarrow q$ in a special case first. Suppose *n*is a positive integer. Let p be the proposition "n = 9" and q be the proposition "$\sqrt{n}$ is an integer". For this pair of propositions, the statement $p \rightarrow q$ is always true; that is, the statement *"n=9"* implies '$\sqrt{n}$ is an integer" is true for all $n \in Z^+$**.** We need to find the truth value of $p \rightarrow q$ for each combination of p false or true with q false or true; so let us see whether we can find a value of *n* that will give us each of the four possibilities. Now p is true when *n=9* and false when $n \neq 9$ while q is true when *n* is a square number and false otherwise. Thus, for example, when n =2, both p and q are false; when *n= 4,* p is false and q is true; and when *n=9,* both p and q are true. However, it is impossible to find a value of *n* that makes p true and q false. This is the ONLY combination of truth values of p and q that would make the implications *false.*

Although we have determined the truth value of $p \rightarrow q$ only for a particular pair of propositions p and q, our conclusions hold in general: for any propositions p and q, $p \rightarrow q$ is false *only* when p is true and q is false. Thus we have the following truth table for $p \rightarrow q$.

p	q	$p \rightarrow q$
0	0	1
0	1	1
1	0	0
1	1	1

Fig 8.5

We can deduce the truth table for $p \leftrightarrow q$ from the table for $p \rightarrow q$, because we have defined $p \leftrightarrow q$ as $(p \rightarrow q) \Lambda (q \rightarrow p)$.

p	q	$p \to q$	$q \to p$	$p \leftrightarrow q$
0	0	1	1	1
0	1	1	0	0
1	0	0	1	0
1	1	1	1	1

Fig 8.6

Thus $p \leftrightarrow q$ is true only when either *both* p and q are true or *both* p and q are false.

Now let us determine the truth sets for $p \to q$ and $p \leftrightarrow q$. We have seen that $p \to q$ is false only when p is true and q is false. The truth set for (p true and q false) is the set *P n Q'*. Hence the truth set for $p \to q$ is *(P nQ')' = P'uQ,* by De Morgan's law. Now *(P n Q')'* is also the truth set for the statement ~(pΛ(~q)), and *P'uQ* is also the truth set for the statement *(~p)vq*. Thus we have found two statements logically equivalent to $p \to q$.

Result 8.2. $p \to q$ = ~(pΛ(~q)) = *(~p)vq*

In Example 8.6, we used truth tables to show that *pΛ(~q)=(~p)vq* and you will see that the truth tables for these statements are the same as the table for $p \to q$ in Fig.8.6, above.

The truth set for $p \leftrightarrow q$ is *(P $\oplus$ Q)',* so we have

Result 8.3. $p \leftrightarrow q = \sim(p \oplus q)$.

Again, this is easily verified by comparing the truth tables for these statements.

Exercise 8.10. Prove that $(\sim p) \leftrightarrow (\sim q)$ is logically equivalent to $p \leftrightarrow q$.

Exercise 8.11. The basis for the method of direct proof by inference is that the statement $[(p \to q)\Lambda(q \to r)] \to (p \to r)$ is always true, that is, it is a tautology. Express the statement $[(p \to q)\Lambda(q \to r)] \to (p \to r)$ in words and then use a truth table to prove that it is a tautology.

Laws of logic

By applying the laws of set algebra to truth sets, we can deduce equivalent laws for manipulating compound statements; these are the laws of logic.

Sets **Propositions**

Commutative Laws:

Sets	Propositions
$P \cap Q=Q \cap P$	$p \wedge q=q \wedge p$
$P \cup Q=Q \cup P$	$p \vee q=q \vee p$

Associative Laws:

Sets	Propositions
$(P \cap Q) \cap R=P \cap (Q \cap R)$	$(p \wedge q)\wedge r=p \wedge (q \wedge r)$
$(P \cup Q) \cup R= P \cup (Q \cup R)$	$(p \vee q) \vee r=p \vee(q \vee r)$

Distributive Laws:

Sets	Propositions
$P \cap (Q \cup R)= (P \cap Q)\cup(P \cap R)$	$p \wedge (q \vee r)= (p \wedge q)\vee (p \wedge r)$
$P \cup (Q \cap R) = (P \cup Q) \cap (P \cup R)$	$p \vee (q \wedge r)= (p \vee q) \wedge (p \vee r)$

De Morgan's Laws

Sets	Propositions
$(P \cap Q)' = P' \cup Q'$	$\sim(p \wedge q) = (\sim p) \vee (\sim q)$
$(P \cup Q)' = P' \cap Q'$	$\sim(p \vee q) = (\sim p) \wedge (\sim q)$

Identity Laws

Sets	Propositions
$P \cap \phi=\phi;\ P \cup\phi=P$	$p \wedge F = F;\ p \vee F=p$
$P \cap U =P;\ P \cup U = U$	$p \wedge T=p;\ p \vee T=T$

Absorption and Complement Laws

Sets	Propositions
$P \cap P=P;\quad P \cup P =P$	$p \wedge p = p;\ p \vee p = p$
$P \cap P'=\phi;\quad P \cup P' =U$	$p \wedge (\sim p) = F;\ p \vee (\sim p) = T$

These laws can be used to prove the logical equivalence of two statements as an alternative to constructing truth tables.

Example 8.8. We prove that $\sim(p \wedge (\sim q))=(\sim p)\vee q$, using the laws of logic.

$\sim(p \wedge(\sim q)) = (\sim p)\vee(\sim(\sim q))$, by De Morgan's Law,

$= (\sim p) \vee q$, by Result 8.1. .

Exercise 8.12. Use the laws of Logic to prove that $(\sim p) \vee (p \wedge q)$ is logically equivalent to $(\sim p) \vee q$.

Our next result establishes the validity of the method of proof by contradiction. Suppose we want to prove that if a given proposition j (called the premise) is true, then the proposition q (called the conclusion) is also true. Recollect that to prove this by the method of contradiction, we first assume that the conclusion q is false and the show that this assumption implies that the premise p is also false. In other words, we are proving $p \rightarrow q$ by showing that $(\sim q) \rightarrow (\sim p)$. We justify this by showing in Result 8.4 that these two statements are logically equivalent.

Result 8.4. $(\sim q) \rightarrow p) = p \rightarrow q$.

Proof. $(\sim q) \rightarrow (\sim p)$ $= \sim t\sim q) \, v(\sim p)$, by Result 8.2,
$= q \, v(\sim p)$, by Result 8.1,
$= (\sim p) \, v \, q$, by the commutative law,
$= p \rightarrow q$, by Result 8.2, again. .

The statement $(\sim q) \rightarrow (\sim p)$ is known as the contra positive of $p \rightarrow q$.

Example 8.10. (a) The contra positive of the statement "If your ticket is drawn, you win a prize" is the statement "If you don't win a prize, then your ticket is not drawn".

(b) The contra positive of the statement "If *n* is prime, then √*n* is irrational" is "If '√*n* is rational, then *n* is not prime". (Note that we have replaced the double negative "not irrational" by "rational".)

Exercise 8.13. Write the contra positive of each of the following statements.
(a) If x is positive, then 'ix is a real number.
(b) If $n > 3$, then $2n > 6$.
(c) If the quadrilateral is a square, then its four sides are equal.
(d) If there is only a finite number of primes, then there is a greatest prime.