CHAPTER 3

FUNCTIONS AND SEQUENCES

Your experience of *functions* in your previous work may well be limited to the polynomial and rational functions and the functions that you have met in courses on calculus, acting on the set of real numbers. In this chapter, we shall give a more general definition of a *function,* and look at some new examples as well as some more familiar ones. We shall also look at functions with particular properties and see how they can be applied to a type of counting problem.

We shall see that a *sequence* can be considered as an example of a function and we introduce a notation for sequences and the idea of a *recurrence relation* in this chapter. This material forms a basis for the discussion of proofs by induction in Chapter 4 of this Study Guide and of finite sums in Chapter 5.

3A FUNCTIONS

Functions are important in all branches of mathematics and in computer science. We shall use them extensively on this course. You will already be familiar with the idea of a function, at least in an intuitive way, but for our present purposes we need to be more precise about what a function is.

Suppose we are given two non-empty sets X and Y and a rule f such that given any element XEX, f assigns to x a unique element $y \in Y$. Then we call f a function of X into Y, written $f:X \to Y$. We shall write the statement **"y** is the element assigned to **x"** in the familiar form

The definition of a function contains two essential conditions that the rule f must satisfy.

(1) $f(x)$ must be defined for *every* $x \in X$.

(ii) Given $x \in X$, then there is only *one* element $y \in Y$ such that $y = f(x)$.

When $f:X \to Y$, the set X is called the domain of f and the set Y is called the codomain of f. We call $f(x)$ the image of x under f and x the pre-image (or ancestor) of $f(x)$. We often say that f maps x onto $f(x)$. The set of all elements of Y that are images of elements of X under f is called the range of f and often denoted by $f(X)$. Thus

$$f(X)=\{ f(x) : x \in X\}.$$

We can illustrate a function with an arrow diagram in which an arrow links each element (or a typical element) in the domain with its image in the codomain.

Exercise 3.1 Which of the diagrams in Fig.3.1 illustrate a function from X into Y? What special features should an arrow diagram possess for it to represent a function?

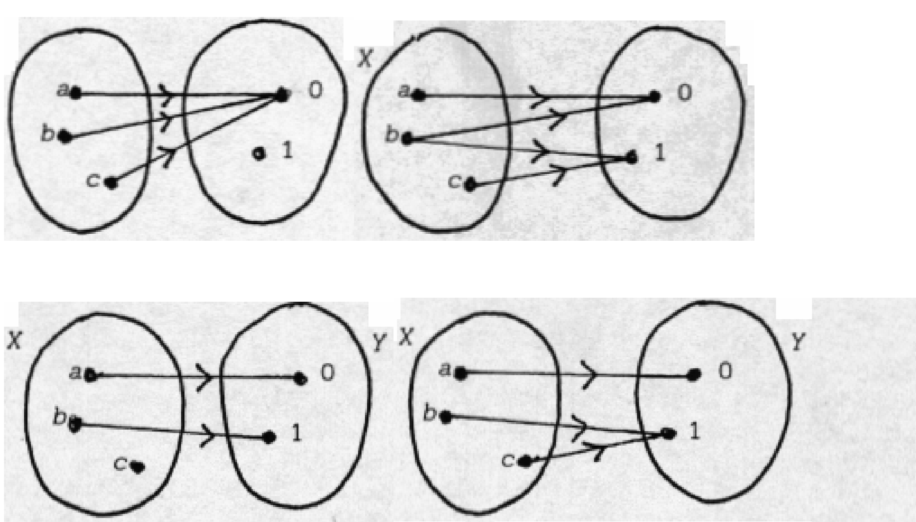

Fig.3. 1

Exercise 3.2 How many different functions can be defined with domain *X=(a,b,c}* and codomain Y={0,1,2,3}?

Although the functions that you have met in mathematics have commonly been denoted by letters such as *f* or a, in a computer language they are usually denoted by a word or string of letters which gives a clue to the operation the function performs. In computing terms, **the domain set** is the set of *inputs* and the **range** is the set of corresponding *outputs.*

Example 3.1 The function INT(x) accepts any real number as input and outputs the integer part of x (that is, the integer obtained by deleting all the figures after the decimal point). Thus, for example, INT($\sqrt{5}$)=2, INT(—25.99)= —25. The domain of INT is the set R of all real numbers .and the range is the set Z of all integers.

Example 3.2 The absolute value of x is denoted by lxi and defined by the rule

$$|x| = \begin{cases} x, & when \quad x \geq 0 \\ -x & when \quad x < 0 \end{cases}$$

Then the rule ABS(x)=|x| defines a function with domain R and range the set $\{x \in R : x \geq 0\}$ (the set of non—negative real numbers).

Example 3.3 Suppose B_3 is the set of all 3-bit binary strings, let S = {a,b,c} and A={0,1}.

(a) Let BIT: P(S) $\rightarrow B_3$ be the rule that assigns to each subset of S its bit vector. So, we input a subset and output its bit vector. For example, BIT({a})=100, BIT({b,c})=011 and BIT(S) =111. Then BIT is a function with domain P(S) (set of inputs) and co domain B_3 (this was specified in the definition of BIT above). In this case, the range (set of outputs) is also B_3.

(b) Let $LAST{:}B_3 \rightarrow A$ be the rule defined by LAST(s) = the last digit of *s.*
Then this function accepts any 3-bit binary string as input and outputs its last digit. So, for example, we have LAST(101)=1 and LAST(110) =0. Thus LAST is a function with domain B_3, co domain *A* and range *A.*

(c) Let SUM:$B_3 \rightarrow$ N be the rule defined by SUM(s) = the number of ones in *s.*
Then SUM accepts any 3-bit binary string as input and outputs the
number of ones in the string. So, for example, we have SUM (000)=O, SUM (OO1)=1, SUM (1O1)=2 and SUM (11l)=3. The rule SUM is a function with domain B_3, co domain N, but range (0,1,2,3).

Notice that in order to specify a function, it is necessary to state the domain and co domain as well as giving the assignment rule. If we change either the domain or the co domain, then we change the function and it is necessary to denote it by a new letter. In the following example, we see the effect of changing the domain.

Example 3.4 Suppose that *B* is the set of all binary strings (with any number of bits). Let $k{:}B \rightarrow N$ be the function defined by

k(s) = the number of ones in *s.*

To appreciate why *k* is a different function from the function SUM defined in Example 3.1(c), consider the problem of trying to draw an arrow diagram to illustrate each of them. This would be a comparatively easy task for SUM but it would not be obvious how to proceed for *k.* Further, the two functions have different *ranges.*

Exercise 3.3 What is the range of the function *k* defined in Example 3.4? Explain why.

In discrete mathematics, we often require an integer as an answer to a problem where perhaps we have calculated the exact solution as a fraction or irrational number. This situation has motivated the definition of the following functions of IR into Z.

Example 3.5
Let **x** $\in$ R Then the rules defined by
FLOOR(x) = greatest integer less than or equal to x , CEIL(x) = the least integer greater than or equal to x are functions, with domain R and range Z, called the floor function and the ceiling function respectively. The value of FLOOR(x) is denoted symbolically by | x | and the value of CEIL(x) by |x | Thus, for example, we know that It is given approximately by 3.14159... . So FLOOR (π) = | π | =3 and CEIL(π) = π =4. Notice that when a $\in$ Z, then [a] =[a]=a.

Exercise 3.4 (a)
Give the values of INT(0.333), INT(-[illegible]), [20/3] , [-13/4], [6], [$\sqrt{2}$], [- $\sqrt{-2}$]

(b) The functions FLOOR(x) and CEIL(x) are obviously closely related to the function INT(x) that we discussed in Example 3.1. Find the set of values of x for which (1) FLOOR(x) = INT(x); (ii) CEIL(x)= INT(x).

Exercise 3.5
Let *n* be an integer. Specify the set {x: x $\in$ R and [x]=n} by using *inequalities* instead of the floor function.

Exercise 3.6 Sketch the graph of the function f : R —> R defined by *f(x)=x.* Superimpose on this graph the graphs of FLOOR(x) and CEIL(x). You will see why [x] and [x] are known as *step* functions.

Exercise 3.7 Give the values of [a/2] and [a/2] in terms of the integer *a*, when (i) *a* is even and (ii) *a* is odd.

Example 3.6
Preparing a supper party, I decide to allow one loaf of bread for every 3 people. Since I cannot buy fractions of a loaf and would prefer to offer my guests too much food than too little, I cater for *n* people by buying [n/3] loaves.

Exercise 3.8 How many of the integers 1,2,3,... ,*n* are multiples of 2? of 3? of *m*, where $m \in Z$ and $1 \leq m \leq n$?

The following family of functions will be more familiar.

A **polynomial** function $f : R \rightarrow R$ is a function of the form

$$f(x) = a_{k+}x^k + a_{k-1}x^{k-1} + ... + a_1x + a_0$$

A polynomial in which *k* is the highest power of x with a non-zero coefficient is said to have **degree** *k*. Polynomial functions of small degree have special names; we often call $f(x)=a_1x+a_0$ (of degree 1), a linear function; $f(x)=a_2x^2+a_1x+a_0$ (of degree 2), a quadratic function; and $f(x)=a_3x^3+a_2x^2+a_1x+a_0$ (of degree 3), a cubic function.

An exponential function $f{:}R \rightarrow R$ is a function of the form

$$f(x)= a.^x$$

where a is a given positive constant. In the branches of mathematics where we use a continuous variable (such as all applications of calculus, for example), the constant *a* is usually the irrational number denoted by *e* that has a value of approximately 2.71828... . However in discrete mathematics and computing, the constant *a* is usually a positive integer and the input values x are also integers. In computing, powers of 2 are particularly important.

Example 3.7
Let EXP:Z $\rightarrow$ Q be the rule that accepts an integer *n* as input and outputs the value EXP (n) $=2^n$.
Then EXP is a function with domain Z, codomain Q and range the set {... ,1/8, 1/4,1/2,1,2,4,8,16,...).

Sequences

A sequence is simply a list as, for example,

(a) 2,5,8,11,14,…..
(b) 5,0.5,0.05,0.005,0.0005, ……
(c) 0,1,1,2,3,5,8,13,21

Formally, a sequence is a function from the set Z^+ into R. The first term in the sequence is called the initial term and is the one we assign to 1, the second term is assigned to 2, the third to 3, and so on. We usually denote the terms of the sequence by a letter with a subscript, thus

$$u_1, u_2, u_3 ....$$

For example, in the sequence (a) given above, the initial term is $u_2 = 2$; then $u_2 = 5$, $u_3 = 8$, and so on.

Sequences are important because they arise naturally in a wide variety of practical situations, whenever a process is repeated and the result recorded. When the process is random as, for example, when the air temperature is recorded at a weather station or a die is rolled, there is no way of predicting for certain what the next term of a sequence will be, however many earlier terms we have knowledge of. There are processes, however, that give rise to sequences where the terms fall into a pattern as, for example, when the value of a sum of money invested at a fixed rate of compound interest is calculated at regular intervals. It is this latter type of sequence, where we can continue the sequence when we know the pattern and the first few terms, that interests us on this course. In this section, our objective is to find a way of expressing the relationship between the terms of this kind of sequence.

Exercise 3.9
What are the next two terms of each of the sequences given at the start of this subsection? Can you suggest two different ways of continuing a sequence that begins 1,2,4,,….?

To be able to continue the intended sequence, you must be given sufficient data to be sure of the relationship between its terms, as the example in Ex.3.9 illustrates.

We have enough terms of the sequence (a) 2,5,8,11,14, ..., to convince us that each term is found by *adding* 3 to the preceding term. So the terms are calculated successively by the rules:

$$u_1 = 2$$
$$u_2 = u_1 + 3 \quad (= 5)$$
$$u._3 = u_2 + 3 \quad (= 8)$$

We can express this relationship between the terms *in general* by $u._{n+1} = u_n + 3$ for all $n \in Z^+$ This is called the **recurrence relation** for this sequence. A sequence for which the recurrence relation is of the form $u_{n+1} = u_n + d$ where *d* is a constant, is known as an arithmetic **progression (A.P.).**

In the sequence (b) 5,0.5,0.05,0.005,0.0005, ..., we obtain each term by *multiplying* the preceding term by 0.1. This time, the terms are calculated successively by the rules:

$$u_1 = 2$$
$$u_2 = (0.1)u_1 \quad (= 0.5)$$
$$u_3 = (0.1)u_2 \quad (= 0.05),...$$

The recurrence relation for this sequence is $u_{n+1} = (0.1)u_n$ for all $n \in Z^+$. A sequence for which the recurrence relation is of the form $u_{n+1} = ru_n$ where *r* is a constant, is called a geometric progression **(G.P.).**

Exercise 3.10

For each of the following sequences,
(i) decide whether it is an A.P. or a G.P. or neither;

(ii) calculate the next two terms of the sequence;
(iii) find a recurrence relation that gives $u_{\sim 1}$ in terms of $u\sim$.

(a) 3,7,11,15, 19,…….. (b) 0,1,3,6,10,……
(c) 2,—6,18,—54,162,……. (d) 1,3,7,15,3,…

The sequence (c), given at the beginning of the subsection, is known as the Fibonacci sequence The terms are called Fibonacci numbers and we shall denote them by F_0 ,F_1 ,F_2,... (note that it is customary to start this sequence at term 0 instead of term 1). The sequence has so many interesting properties that it has fascinated mathematicians for centuries. Recently a number of applications have been found to computer science.

We hope that you decided in Ex.3.9, that the next two terms of sequence Cc) are 34 and 55. In this case you will have realized that each term is the sum of the previous two. So, starting from the initial terms $F_0=0$, $F_1=1$, the terms are calculated successively by the rules:

$F_2=F_0 \div F_1$ (=0+1=1),
$F_3 = F_1 + F_2$ (= 1 + 1 = 2),
$F_4 = F_2 + F_3$ (=1+2=3),
$F_5 = F_3 + F_4$ (=2+3=5)

The recurrence relation this time is given by $F_{n+2} = F_{n+1} + F_n$ where $n \geq O$ Notice that this time we need knowledge of two initial terms, F_0 and F_1, in order to use the recurrence relation to calculate successive terms.

Exercise 3.11
Continue the Fibonacci sequence as far as F_{10} Which terms give even integers? Can you give an explanation for this pattern?

Exercise 3.12
Make a table to show the value of ui.,, for n=1,2,3,4,5, for the sequences determined by each of the following recurrence relations.

(a) $u_{n+1} = 5u_n + 2, \quad u_1 = 0$ (b) $u_{n+1} = u_n + 2^{n+1}, \quad u_1 = 2$

(b) $u_{n+1} = nu_n \quad u_1 = 1$ (d) $u._{n+2} = u_{n+1} + 2u_n, u_1 = 1 \quad u_2 = 2$

3B FUNCTIONS WITH SPECIAL PROPERTIES

For a given a function, there are two questions that we often wish to ask.
(1) Are the range and the co domain equal?
(2) Do distinct elements in the domain always have distinct images?

(1) A function that has the property that its range equals its co domain is called an onto function. We have already met examples of such functions. For instance, the functions BIT : P(S)$\rightarrow$B3 and LAST:B_3 $\rightarrow$A defined in Example 3.3 are both onto functions.

Test for the onto property. To test whether the function f:X$\rightarrow$Y is an onto function, we let y$\in$Y and see if it is possible to find x$\in$X such that f(x) = y. If we can do this, whatever the value of y, then f is onto and we say that f maps X onto Y. However, if we can find an example of an element y$\in$Y that has no pre-image in X, then f is not onto.

Example 3.8 (a)
Consider the function FLOOR:R$\rightarrow$Z. Given any integer n$\in$Z, we can always find (at least one) real number x $\in$R such that FLOOR(x)=n, since we could have x=n, for example. Thus FLOOR is an onto function and we say FLOOR maps R onto Z.
(b) Let EXP:Z$\rightarrow$D be defined by EXP(a)=2^n. In this case, not every y$\in$Q arises as an image under EXP. For example, when y=3, we cannot find a$\in$Z such that EXP(a) = y.

Exercise 3.13
Can either of the functions illustrated in Fig.3.1 (a) and (d) be described as onto? What special features does an arrow diagram have if it represents an onto function?

(2) A function is called one-to-one if every pair of distinct elements in the domain have distinct images in the co domain.

Example 3.9 (a)
Let EXP:Z $\rightarrow$ D be defined by EXP(a)=2^a. Let a,b be any pair of distinct elements in the domain set Z. Then a $\neq$ b implies that $2^a \neq 2^b$ Hence a and b have distinct images and so EXP is one-to-one.

(b) Consider the function FLOOR:R $\rightarrow$ Z. In this case, it is easy to find a pair of distinct elements of that have, the same image.
For example, FLOOR(1.1)=FLOOR(1)=1. Hence FLOOR is not a one-to-one function.

Exercise 3.14
Show that the function LAST $B_3 \rightarrow$ A (define in Example 3.3(b)) is not one-to one.

Exercise 3.15

Let $S = \{x_1, x_2 ....x_n\}$ (n $\geq$ 2) and the function SIZE:P(S) $\rightarrow$ N be defined by SIZE(x) = $|x|$, for every subset $X \subseteq S$. What is the range of SIZE? Is SIZE onto? Is SIZE ever one-to-one?

When a function $f : X \rightarrow Y$ is both one-to-one and onto, we say that f is a one-to-one correspondence from X onto Y.

Example 3.10

In Example 3.3, the function BIT: P(S) $\rightarrow B_3$ is one to one because no two different subsets have the same bit-vector. It is also an onto function and hence it is one-to-one correspondence, because it is not one-to-one correspondence, because it is not one-to-one. The function EXP:Z $\rightarrow$ Q defined by EXP (a)=2^a is also not a one-to-one correspondence, because it is not onto.

Inverse functions:

Let f: X $\rightarrow$ Y be a function, Suppose we define a rule v with domain Y and Co domain X by

$$v(y) = x \quad \text{if and only if} \quad f(x) = y$$

Exercise 3.16

Draw an arrow diagram to illustrate the function f and the rule v given by each of the following tables, where X = {a,b.c},
Y={0,1}, Z={0,1,2,3} and W={0,1,2}. Whaen is v a function?

(a) f: X $\rightarrow$ Y

x	a	b	c
f(x)	1	0	1

(b) f :X $\rightarrow$ Z

x	a	b	c
f(x)	3	0	1

(c) f: X $\rightarrow$ W

x	a	b	c
f(x)	2	0	1

You will see that *v* is the rule that starts from the co domain of *f* and maps each element back into its pre-image. In general *v* will not be a function, but when *f* is a *one-to-one correspondence,* then *v* satisfies the conditions for a function. This is proved in the following theorem.

Theorem 3.1. *Let f: X4Y be a one-to-one correspondence. Then the rule v with domain Y and co domain X defined by*
v(y)=x if and only if f(x)=y is a function from Y onto X.

Proof: Since f is onto, every element of Y has a pre-image in X and so v(y) is defined for every element of Y. Further, since f is one-to-one, no two distinct elements of Y have the same pre-image in

X and hence there is only one element $x \in X$ such that v(y)=x. Thus v satisfies both the conditions for it to be a function from Y into X. .

In this case, *v* is called the inverse function of *f;* we write *v* as f^{-1}*:Y-~X* and the function *f* is said to be invertible.

Exercise 3.17
Decide for each of the following functions whether it is one—to—one, onto or neither. Say which of these functions are *invertible* and define the inverse of each invertible function.

(a) $f : Z \to Z$ defined by f (a) = a + 3

(b) $f : Z^+ \to Z^+$ defined by f(n)=n+3

(c) $f : R \to R$ defined by f(x) = 2x + 5

(c) $f : Z \to Z$ defined by f(a) = 2a +5

(e) ABS : R $\to$ R defined by ABS(x) = | x |

(f) $f : ZxZ \to Z$ defined by f(a,b) = a +b

(g) $f : Z \to Z$ defined by $f(n) = \begin{cases} n+2 & when \quad n \quad is \quad even \\ 1-2n & when \quad n \quad is \quad odd \end{cases}$

Exponential and logarithmic functions

These functions are of great importance in computer science, particularly when we come to determine the number of steps in the implementation of an algorithm. Let a be a given fixed positive real number. Suppose that the function EXP:IR-~R~ is defined by EXP(x)=ax. Then the function EXP is both one-to-one and onto and hence is invertible. Its inverse function is called the logarithm of x to the base a and is denoted by $\log_a x$. Thus $\log_a x$ is defined by

$$y=\log_a x \text{ when } x=a^Y.$$

The domain of $\log_a x$ is $R^+$and its codomain is R.

Exercise 3.18

Show how to evaluate the following logs directly from the definition without using a calculator.

$\log_2 16$, $\log_2 (1/8)$, $\log_8 2$, $\log_2 (\log_2 256)$.

Functions and counting

You will be familiar with the idea of a one-to-one correspondence in an intuitive way. In fact, whenever you count a finite set of *n* objects you are mentally matching them with the numbers 1,2,... ,*n*. In other words, you are putting the set into one-to—one correspondence with the set {1,2,.. .*n}*. It is also intuitively obvious that if we can find a way of matching the elements of two finite sets, then these sets must have the same cardinality. We used this idea in the proof of Theorem 1.2, where we matched the elements of the set P(S) with those of B_3. We can express this counting idea more precisely in terms of functions.

Let X be a finite set and *f:* $X \to Y$ be a *one-to-one* function of X into some set *Y*. Then every element in the *range f* corresponds to exactly one element in the domain set X. Hence |f(x)| = |x|. Now if we also require *f* to be an *onto* function, then the range of *f* is the co domain *Y* and hence |Y| = |f(x)| =|x|. This gives the following counting principle.

The Correspondence Principle. *Let X, Y be finite sets and suppose we can find* a *one-to-one correspondence* *f:* $X \to Y$. *Then* | x | = | y |

Suppose, on the other hand, that we wish to define a function *f:* $X \to Y$, where X and *Y* are finite sets with |X| >|Y|. Then there must be (at least) two distinct elements of X that have the same image. Use this simple idea to solve the following problems.

Exercise 3.19

Show that in any group of 13 people, there will be at least two who have their birthdays in the same month.

Exercise 3.20

Suppose a bag contains twelve balls, two in each of six different colours. The balls are drawn at random from the bag. How many balls need to be drawn to be sure of finding two balls of the same colour?