CHAPTER 2
SETS AND BINARY OPERATIONS

2A OPERATIONS ON SETS

In this section, we suppose that A,B,C are subsets of some universal set U. We shall define some operations that can be performed on these subsets to yield other subsets. We shall assume that you are familiar with the pictorial representation of sets known as **Venn diagrams.** If not, you will find them described in most books on Discrete Mathematics or introductory Algebra. We shall introduce an alternative and equivalent way defining a subset using a **membership** table and show how these and Venn diagrams can be used to verify relations between sets.

Complement of *A:* $A' = \{x : x \in A\}$.

Notice that U'=0 and ϕ'=U.

(Common alternative notations found in textbooks for the complement of *A are* -A and A.)

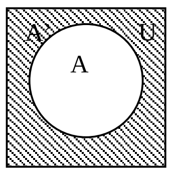


Fig. 2. 1.

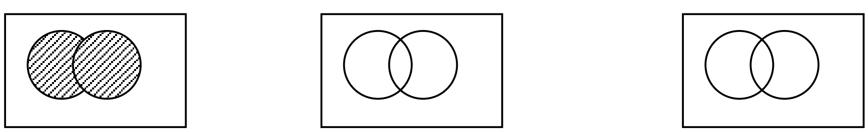
The other operations that we shall define combine two subsets and are in consequence known as **binary operations.**

Union of A **and** B: $A \cup B = \{x : x \in A \text{ or } x \in B \text{ (or both)}\}$

Intersection of *A* **and** B: $A \cap B = \{x : x \in A \text{ and } x \in B\}$.

Note that when $A \cap B = \phi$, the sets A and B are said to be **disjoint.**

Difference of A and *B:* $A - B = \{x : x \in A \text{ and } x \notin B\}$; $B - A = \{x : x \in B, x \notin B\}$.



$A \cup B$ $A \cap B$ A-B

Fig. 2.2.

Notice that in drawing a Venn diagram to illustrate two sets in general, we depict the sets as overlapping, as shown in fig.2.2. Drawn in this way, the boundaries of the sets partition the whole area (representing U) into four discrete regions. We are not saying that in every example, there are necessarily elements in each of these regions; in any particular example it may be happen that one or more of the region is empty. In particular, if A and B are disjoint, then the region representing $A \cap B$ will be empty.

Example: 2.1:
Suppose U.=Z, A={1,3,5,7,9}, B={2,4,5,6,7,8} then
$A \cup B$ = {1,2,3,4,5,6,7,8,9}; $A \cap B$ = {5, 7}; A-B={1,3,9} and B-A={2,4,6,8}

Exercise 2.1:
Let A be any subset of a universal set U. Find another way of describing the subsets (i) U – A (ii) A – U Symmetric difference of A and B : $A \oplus B = \{ x : x \in A \text{ or } x \in B, \text{ but not both}\}$
$= (A- B) \cup (B – A)$
Exercise 2.2: Draw a Venn diagram to illustrate $A \oplus B$

Our first two theorems follow directly from the definitions of union, intersection and complement. You can easily verify them from the Venn diagram in figures 2.1 and 2.2
Theorem 2.1:
Let A, B be any two sets. Then
(i) $A \cap B \subseteq A \subseteq A \cup B$ and $A \cap B \subseteq B \subseteq A \cup B$
Moreover, if $A \subseteq B$, then
(ii) $A \cap B = A$ and $A \cup B = B$

Notice that as special case of Theorem 2.1(ii), we have
$A \cap A = A$ and $A \cup A = A$

Theorem 2.2:
(Identity and Complement laws). Let A be a subset ofa universal set U. then
(i) $A \cap \phi = \phi$ and $A \cup \phi = A$
(ii) $A \cap U = A$ and $A \cup U = U$
(iii) $A \cap A' = \phi$ and $A \cup A' = U$

Examples of binary operation on the real numbers are addition, multiplication and subtraction (division is a binary operation on the non-zero reals). Addition and multiplication obey certain rules that are familiar that we tend to take them foe granted. For examples, we know that for any two real numbers x, y.
$x + y = y + x$ and $xy = yx$
We say that the operations of addition and multiplication are commutative, because we can commute (or interchange) the relative positions of x and y without changing the value of the sum or product. On the other hand, subtraction is not a commutative operation because the statement "x – y = "y-x" is not a true for all values of x and y. You will encounter other examples of non-commutative operations, such as matrix multiplication and symmetry operations geometrical shapes, in your mathematics courses.

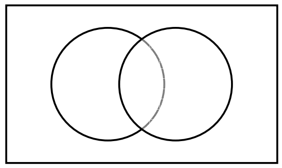
In the case of the set operations union and intersection, however, it is clear from their definitions, and also from the Venn digram in fig 2.2 that A u B = B u A and A n B = B n A, for all sets A, B. So we have:

Theorem 2.3: (Commutative laws). For any two subsets A and B of a universal set U
(i) A n B = B nA (ii) A u B = B u A

Before considering binary operations on more than two sets, it is useful to develop an alternative method to Venn diagrams for illustrating and verifying our results. This can be done by constructing a **membership table** for each of the sets concerned. Suppose that *A*,*B* are subsets of a universal set *'LI* and let x ∈ *'U* Then just one of the following four possibilities must be true.

(a) $x \notin A$, $x \notin B$; (b) $x \notin A$, $x \in B$;
(c) $x \in A$ $x \notin B$ (d) $x \in A$, $x \in B$.

These four possibilities are illustrated by the Venn diagram shown in fig.2.3 below. The regions are labeled so that if (a**)** is true, then x belongs the region labeled a, and so on.



We now compile a table in which each row corresponds to one of the four statements (a)—(d) concerning the element x or, equivalently, to the statement that x belongs to one of the corresponding regions a,b,c,d in the Venn diagram. Each column represents a given set, with the first two columns representing A and B respectively. In the column corresponding to a given set X, we enter 1 in each row for which the statement XEX is true and 0 in each row for which XEX is false. Thus, for example, the set AuB contains all elements x for which just statements (b),(c),(d) are true. Hence we enter a 1 in the rows corresponding to regions b,c,d and 0 in the row corresponding to a. The table in fig.2.4 shows the membership of the sets A,B,Ar~B andAuB.

A	*B*	*AnB*	*AuB*
0	0	0	0
0	1	0	1
1	0	0	1
1	1	1	1

Fig.2.4.

Notice that once we have observed the correspondence between the rows and the regions of the Venn diagram, it is unnecessary to continue recording the letters *a,b,c,d* beside the rows. The entries in the *A* and *B* columns in each row clearly tell us which region corresponds to that row. For example, entries 0,1 in the *A*,*B* columns tell us that this row corresponds to the region where the elements are in *B* but not in *A.*

Exercise 2.3. The subset of *U* corresponding to the region labelled a in the Venn diagram in fig.2.3 can be described as $\{x: x \in A,\ x \notin B\}$=*A'nB'*. Describe the subsets corresponding to each of the regions *b,c,d* as the intersection of two subsets.

Exercise 2.4. (a) Compile a membership table for each of the subsets
A-B, *B-A* and $A \oplus B$.
(b) Let U ={1,2,3}. Find an example of subsets *A*,*B* for which $A\text{-}B \neq B\text{-}A$.

We can use membership tables to prove the following two laws relating to set complements, due to the nineteenth century mathematician Augustus De Morgan.

Theorem 2.4. (De Morgan's laws).
Let A,B be subsets of a universal set U then
(i) *(A n B)' = A'* u *B';* (ii) (A u B)'=A'n B'

We compile a membership table to prove i). We need to construct and compare columns for *(AnB)'* and A'uB'. We first construct columns for each of the constituent sets of these expressions, namely *A',B'* and *AnB.* Then we obtain the column for *A' u B'* from *A'* and *B'* and the column for *(AnB)'* from *AnB.*

A	*B*	*A'*	*B'*	*A'* u *B'*	*A n B*	(A n *B)'*
0	0	1	1	1	0	1
0	1	1	0	1	0	1
1	0	0	1	1	0	1
1	1	0	0	0	1	0

Since the columns corresponding to *(AnB)'* and A'uB' are identical, we know that if $x \in$ *(A n B)'*, then $x \in$ A'u B' and conversely. Pictorially, it means that the Venn diagram for each of these subsets will comprise precisely the same regions of U. Hence these sets are equal.

Exercise 2.5. Verify the result of Theorem 2.4 (i) by drawing a Venn diagram to show each of the sets *(AnB)'* and A'uB'.

Exercise 2.6. Use membership tables to prove Theorem 2.4(u).

Exercise 2.7. Use a Venn diagram or membership table to prove that
(A')'= A. Using this result and De Morgan's laws, prove that
(A'n B')'=A u B.

We next consider combinations of more than two sets. Figure 2.5 shows how three sets can be represented in the most general way by a Venn diagram. You will see that their boundaries subdivide the area representing the universal set into *eight* discrete regions.

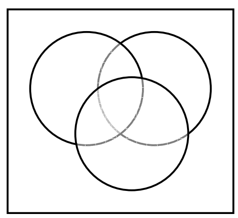


Fig 2.5

Each of these regions corresponds to one of the following eight statements:
(a) $x \notin A, x \notin B, x \notin C$; (b) $x \notin A, x \notin B, x \in C$; (c) $x \notin A, x \in B, x \notin G$; (d) $x \notin A, x \in B, x \in C$;
(e) $x \in A, x \notin B, x \notin C$; (f) $x \in A, x \notin B, x \in C$; (g) $x \in A, x \in B, x \notin C$; (h) $x \in A, x \in B, x \in C$;

A membership table for a subset expressed as a combination of three sets *A,B,C* must therefore have ***eight*** rows, one corresponding to each of the eight statements (a) to (h) above. In fig.2.6, we give the membership table for each of the sets *A,B,C.* Check that each row in the table corresponds to the region of the Venn diagram labeled with the same letter in fig.2.5.

	A	B	C
a	0	0	0
b	0	0	1
c	0	1	0
d	0	1	1
e	1	0	0
f	1	0	1
g	1	1	0
h	1	1	1

Fig.2.6.

Again, we do not need to label the rows in future, because the entries in the columns for *A,B,C* tell us the subset (or region of the Venn diagram) that the row refers to. For example, the entry 101 tells us that the row refers to the subset *AnB'nC.*

Exercise 2.8. Shade on a copy of the Venn diagram shown in fig.2.5, the regions corresponding to each of the sets X,Y,Z defined by the membership tables in fig.2.7. What is the relation (if any) between sets X and ***Z?***

A	*B*	*C*	*X*	*Y*	*Z*
0	*0*	*0*	*0*	*1*	*0*
0	*0*	*1*	*0*	*1*	*1*
0	*1*	*0*	*0*	*0*	*1*
0	*1*	*1*	*1*	*0*	*1*
1	*0*	*0*	*0*	*1*	*0*
1	*0*	*1*	*0*	*1*	*0*
1	*1*	*0*	*1*	*0*	*1*
1	*1*	*1*	*1*	*0*	*1*

Fig.2.7.

We now give two important laws concerned with the use of *brackets* in expressions involving three or more sets. The first of these is the associative law, which says that when we want to find AuBuC or *AnBnC,* we do *not* need brackets to tell us which pair of sets we should combine first. This is because whichever way we put the brackets in, the result is the same set.

Theorem 2.5. (Associative laws). *For any three subsets A,B,C of* a *universal set U*

(i) *(A n B) n C = A n(B n C);*
(ii) (A u *B)* u *C* = A u *(B* u ***C).***

We can prove this law either by Venn diagrams or by membership tables. We give a proof of part (i) using membership tables. We start with the columns for *A,B,C,* as shown in fig.2.6. (The order of the rows does not matter, but it is a good idea to develop a systematic way of listing them, so that you can be sure that you have included all eight different cases.) We next construct the columns for the intermediate steps, in this case for the sets An B and *B n C.* We use these with columns ***C*** and *A* respectively to obtain the columns for *(A n B) n C* and *A n (B n C).*

A	B	C	A n B	(An B)n C	B n C	An(BnC)
0	0	0	0	0	0	0
0	0	1	0	0	0	0
0	1	0	0	0	0	0
0	1	1	0	0	1	0
1	0	0	0	0	0	0
1	0	1	0	0	0	0
1	1	0	1	0	0	0
1	1	1	1	1	1	1

Since the columns for (An B) n C and *(A n B) nC* have identical entries, these two sets are equal.

Exercise 2.9. (a) On a Venn diagram of the sets *A,B,C,* identify the regions corresponding to each of the columns in the membership table above for the proof that *A n (B n C)=(A n B) n C.*

(b) Either by drawing suitable Venn diagrams or by constructing membership tables, prove part (ii) of Theorem 2.5.

Exercise 2.10.

The subset corresponding to row *f* in the membership table in fig.2.6 can be described as
{x : x $\in$ A, x $\notin$ B, x $\in$ C}= A n B'n C. Describe the subsets corresponding to each of the other rows of this table as the intersection of three subsets.

Binary operations that satisfy the associative law are said to be associative. Not all operations are associative, however.

Exercise 2.11.

Let U = {1,2,3}. Find an example of subsets *A,B,C* to show that set difference is not an associative operation.
The second law that deals with the use of brackets is the distributive law. This law relates to expressions involving two *different* binary operations. It gives a rule for expanding a bracket or, equivalently, for factorizing an expression.

Theorem 2.6. (Distributive laws). *For any three subsets A,B,C of* a *universal set* 'LI,
(i) *A* n *(B u C)* = (A n *B)* u (A n *C);*
(ii) A u *(B n C)* = (A u *B)* n (A u *C).*

Exercise 2.12.
(a) Use membership tables to verify Theorem 2.6 (i).
(b) Identify the regions of the Venn diagram shown in fig.2.5 that
comprise the sets (i) Au(B nC); (ii) (AuB) n (AuC). Hence prove Theorem 2.6 (ii).

Exercise 2.13.

Express as neatly as possible using set binary operations the sets X,Y,Z defined in Exercise 2.8.

2B SETS AND COUNTING

In Chapter 1, we defined the **size** or **cardinality** of a finite set S to be the number of elements in S and denoted this by | S | In this section, we shall find a way of counting the number of elements in the union of two or more sets.

Example 2.2. Let the universal set *U* = Z and suppose *A* = {1,2,3},
B = {0,4,5} and *C* = {3,4,5,6}.
(a) *AuB* = {0,1,2,3,4,5}, so that |AuB| = *6* = IAI+IBI.
(b) *AuC* = {1,2,3,4,5,6}, so that |AuC| = 6. But in this case,
|A| + |C| = 3+4 = 7.
(c) BuG = {0,3,4,5,6}, so that |B u C| = 5. But in this case,
|B| + |C| = 3+4 = 7.

We see that in Example 2.2 (a), we have the simple formula |AuB| =|A| + |B|, *only* because the sets *A* and *B* are *disjoint.* This is an example of our next counting principle. Before stating this, we need two definitions. A collection of subsets A_1 ,A_2,. . . ,A_n is called pairwise disjoint if no pair of the subsets has any element in common, that is A_i *n* A_j~=ø, if *i_j*.

Example 2.3.(a)
In an ordinary pack of 52 playing cards, let *C* denote the set of cards in the suit clubs, *D* denote the set in the suit diamonds, *H* denote the set in the suit hearts and S denote the set in the suit spades. Then the

sets *C,D,H,S* are *pairwise disjoint.*

(b) Let A_k be the set of natural numbers of which the last digit is a *k*, k=0,1,... ,9, so that A_0={0,10,20,. **..},** A_1={1,11,21,...), and so on. Then the sets $A_0, A_1, \ldots, A_9$ are *pairwise disjoint.*

Now you will see that the union of the sets of cards in Example 2.3(a), gives the complete pack. We say that the sets *C,D,H,S* **partition** the complete set of playing cards. Formally, a partition of a set X is a collection of non-empty, pairwise disjoint subsets whose union is X.

Example 2.4. The sets $A_0, A_1, \ldots, A_9$ of Example 2.3(b), form a partition of the set of natural numbers. We could partition IN into many other different collections of pairwise disjoint subsets as, for example, the set of even integers and the set of odd integers.

The Addition Principle. *Suppose that* $X_1, X_2, \ldots, X\sim$ *is a partition of a finite set X. If* X_1 *contains* m_1 *elements,* X_2 *contains* m_2 *elements, and so on, then the size of X is* $m_1+m_2.\ldots+m_n$

We now consider examples where we don't have disjoint subsets. In Example 2.2 (b), the sets *A* and *C* and overlap, with *AnC={3}.* So when we add the sizes of *A* and *C*, the element 3 is counted *twice*, once in Al
and once in |C|. Thus |AuC| = |A| + |C| -1. Similarly, in Example 2.2 (c), the sets *B* and *C* overlap, with |B n C|={4,5}. Thus when we add the sizes of *B* and *C*, both the elements 4 and 5 are counted twice, and hence |BuC|=|B|+|C|-2. These examples suggest the following counting theorem.

Theorem 2.7. *Let A and B be finite subsets of a universal set U. Then*

$$|A \cup B| = |A| + |B| - |A \cap B|. \qquad \text{(I)}$$

Proof. We check first that each element of AuB is counted once on the right hand side of formula (I). If x $\in$ AUB, then x belongs either (i) to just one of *A,B* or (ii) to both the subsets *A, B.*

In case (1), $x \notin A \cap B$ and hence x is counted once in |A|+ |B| and 0 times in |*AnB*|, so it is counted once altogether. In case (ii), x $\in$ AnB and hence x is counted twice in |A|+|B| and once in |AnB|. Hence x is counted 2-1=1 time altogether. Thus each element of *AuB* has been counted once on the right hand side of formula (I).

However, if $x \notin AuB$, then x contributes 0 to each of |*A*|, |B| and |AnB|. Hence the right hand side of equation (I) counts exactly the number of elements in |AuB|. •

Exercise 2.14. How many 6-bit binary strings either begin or end with a 0?

Exercise 2.15. A sociologist conducted a survey of 143 students taking A—levels concerning their intentions regarding higher education. Two of the questions asked were: (a) Did either of your parents study for a degree at a university or polytechnic? (b) Do you hope to study for a degree at a university or polytechnic? It was reported that 23 students answered *"Yes"* to (a), 95 answered "Yes" to (b), while 44 answered "No" to both questions. Assuming that all the students in the survey answered either "Yes" or "No" to both questions, how many students who said they hoped to study for a degree at a university or polytechnic had a parent who had done so?

We can generalize Theorem 2.7 to give the following result for the cardinality of the union of three sets.

Theorem 2.8. *Let A,B,C be finite subsets of a universal set U. Then*

$$|AuBuC| = |A|+|B|+|C| - (|AnB|+|AnC|+|BnC|) \div |A \cap B \cap C|. \qquad \text{(II)}$$

This theorem can be proved by a counting method, similar to the way in which we proved Theorem 2.7, but the proof is not required for this course.

Example 2.5. Let *U* = *Z*, A={1,2,3,4}, B={1,3,5,7} and C={3,4,5,6}. We verify Theorem 2.8 for |AuBuC|

First note that *AuBuC={1,2,3,4,5,6,7}.* Also, *AnB={1,3}, A nC={3,4}, B nC={3,5}* and *A n B n C={3}.* Hence
|A|+|B|+|C| - (|AnB|+|AnC|+|BnC|) + |*AnBnC*| = *4+4+4* — (2+2+2) + 1 = 7
=|*AuBuC*|, as expected. .

The two theorems we have given in this section are special cases of a general counting theorem known as the Principle of Inclusion-Exclusion. This name derives from the fact that to count the number of elements in the union of several subsets, we alternately include all the elements and then exclude those that have been counted twice. The general form of the Principle of Inclusion—Exclusion is discussed in the references given below, but applications to counting the elements in the union of more than three sets will not be expected in this half—unit. The next example illustrates how Theorem 2.8 is applied.

Example 2.6. Find the number of integers between 1 and 99 (inclusive) that are not divisible by 2, 3 or 5.
We tackle this kind of problem by first counting the number of such integers that *are* divisible by one or more of 2, 3 or 5; then we know that the remaining integers in the set L1={1,2,. . .,*99}* are *not* divisible by any of 2, 3 or 5.

Let *A*, *B* and *C* denote the subsets of *U* containing all the integers divisible by 2, 3 and 5 respectively. Then *AnB, AnC* and *BnC* contain respectively all the integers divisible by 6, 10 and 15, and *AnBnC* those divisible by 30. Hence
A={2,4,6,...,98}, so |A|= 49; *B={3,6,9,. ..,99},* so |B| = 33;
C={5,1O,15,. . . ,95}, so |C|=19; AnB={6,12,18,. .. ,96}, so |AnB|=16; AnC={1O,2O 90}, so |AnC|=9; BnC={15,30,. . .90}, so |BnC|=6;
AnBnC={3O, 60,90 }, so |*AnBnC*| **=3.**

Now *AuBuC* is the subset of integers divisible by one or more of 2, 3 or 5, and Theorem 2.8 gives:
|AuBuC| = |A|+ |B|+ |C| - (|AnB|+|AnC|+|BnC|) + |*AnBnC*|
= 49+33+19 **—** (16+9+6) + **3**
= 73.
Hence the number of integers in *IL* not divisible by 2, 3 or 5 is
99-73=26. .

It is also possible to solve these counting problems by putting the data onto a Venn diagram, rather than by using the Principle of Inclusion—Exclusion. We illustrate how we would do this for the data in Example 2.6. Using the Venn diagram with the notation shown in fig.2.5, we first enter |*AnBnC|=3* in region *h* and then work outwards. Since |AnB|=16, the total in region g is 16—3=13; similarly, we put 9—3=6 in region *f* and 6-3=3 in region *d.* Then the number in region *e* is 49-(3+13+6)27; similarly, we put 14 in *c* and 7 in *b.* Then, since the regions b,c,.. . ,*h* partition *AuBuC,* we have |AuBuC|=3+13÷6+3+27+14+7=73, by the Addition Principle.

Exercise 2.16. In a survey, 180 school children were asked which, if any, of the T.V. soaps *Neighbours, Coronation Street* and *East-Enders* they regularly watched. It was claimed that 123 watch *Neighbours,* 69 watch *Coronation Street* and 95 watch *East Enders;* of these, 45 watch both *Neighbours* and *Coronation Street*, 65 watch both *Neighbours* and *East-Enders,* 42 watch both *Coronation Street* and *East-Enders,* while 30 children claimed to watch all three series. How many claimed to watch none of them?